Mathematical Modeling: Teaching the Open-ended Application of Mathematics

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To my parents, who are wonderful models for how to live a life.

To my high school teachers Stewart Galanor and Antonia Stone, who taught me well and then put me to work.

To my wife, Emily, for all of her encouragement.

About This Book

The curriculum described here consists of five content units and one unit on pedagogy. The material can serve as the backbone of a modeling course for students with some algebra skills or can be woven into a series of grade seven through twelve courses as a modeling strand. These materials have been taught as a yearlong course to mixed groups of students with experience ranging from Algebra and Geometry to those with a year or more of Calculus. Mathematical Modeling is designed as a coherent course that will challenge students throughout this spectrum, expose students to the breadth, depth, utility, and beauty of mathematics, and train students to be able to use mathematics in new and fruitful ways. All students leaving this course are prepared to study many mathematics, science, and social science subjects with a new set of tools and perspectives at their disposal.

The documents that follow provide a framework for thinking about the teaching and learning of mathematical modeling. They do not provide day-to-day details and homework assignments for the entire year, but they do provide in-depth descriptions of some of the main content modules, their associated class activities and project assignments, and the connections between, and reasons for, the topics investigated. In some cases, more activities are provided for a given topic than would be used in a given class in a given year. For me, the choice of activities has been dictated by the types of questions that the students have raised, their apparent mastery or need for further exploration, and my desire to experiment with a new approach. The course has changed considerably each year I have taught it, but the goals have remained relatively stable. At any particular time, different sections may be studying different topics or pursuing separate activities in response to students' questions and interests.

This course was designed in response to several concerns and to address several objectives. The first concern was that traditional mathematics courses, such as Precalculus, are too often about delayed intellectual gratification; they are *pre*-paration for some other experience rather than courses about a real subject with its own history and merit. Too many students and adults have told me that they never got to the course that was finally supposed to teach them the nature or utility of mathematics.

The second concern was that mathematics be a tool that students use to understand their world. Active users of mathematics must be able to approach and solve unfamiliar problems as independent from expert guidance as possible. Because mathematics is a vast and always-growing discipline, we cannot teach our students all that they will need to know. Thus, our greatest challenge is to teach them how to learn mathematics without us.

A third concern, often cited by the National Council of Teachers of Mathematics and the Mathematical Association of America, is that students are reaching college as rigid thinkers with underdeveloped intuitions about quantities, infinite processes, or functions and with little persistence or openness to experimentation. Many of the skills that our students learn are disconnected and inert for them. Mathematics classes must devote a significant portion of their time and activities to addressing this concern. The higher order habits of mind that require creativity and flexibility do not materialize out of long-term exposure to studying and practicing specific technical skills. The activities in this curriculum are created to nurture careful, critical analysis, original problem posing, and the patience essential to success with challenging endeavors. Lastly, American mathematics courses have been criticized for being “a mile wide and an inch deep.” (TIMMS report) As the course has evolved, fewer mathematics topics have been emphasized in ever-greater depth. Paraphrasing one mathematics educator: short run-ups only lead to short jumps, long run-ups lead to big intellectual jumps.

I have not attempted to present a complete curriculum with instructions on how each lesson should proceed. I have mixed traditional lesson plans, dialogues, and descriptions in order to convey best the flavor and substance of the student and teacher experience. I believe the most effective courses reflect teachers’ own intellectual passions. I hope you will borrow, adapt, and blend what I present here with your own best practice. Folk singer Woody Guthrie once claimed that “plagiarism is the basis of all culture” and criticized a fellow folk singer by noting that “he only steals from me, but I steal from everybody.” In that spirit, I offer not a single, “teacher-proof” package, but a teacher-dependent resource.

What is Mathematical Modeling?

Mathematical modeling is the process of using mathematics to study a question from outside the field of mathematics. A mathematical model is a representation of a particular phenomenon using structures such as graphs, equations, or algorithms. This course gives students practice formulating interesting questions from fields such as science, entertainment, politics, or design. It teaches the specific skills used in creating and interpreting mathematical models. These models, in turn, produce new understandings about the original settings of interest and help students answer the questions that they have posed.

The culminating activity of this curriculum (as described in the final unit of the book) is a several-weeks-long group project for which students pose their own questions and develop an original model. Sample questions that students have explored include:

How does dirt affect the melting of snow mounds?

What harvesting guidelines will protect a lobster population in the face of uncontrollable natural variations in the environment?

How can delegates to the United Nations be seated to maximize harmony?

How should a town structure the penalties for speeding tickets in order to generate the greatest revenues?

What arrangement of ceiling lights provides the most even illumination of a room?

As these questions suggest, the goals for a modeling endeavor are varied. Modelers seek to gain understanding, predict outcomes, make decisions, and develop designs. The snow group, inspired by the ever-dirtier mounds piling up during a snowy winter, simply wanted to understand what they were observing. They knew that the process of identifying variables, creating representations, working with those abstractions to generate new information, and determining the significance of that information to the original question would help them toward that end. The lobster group worried that harvesting regulations might be set to leave the minimum number of lobsters needed to replenish the stock. They developed a model for predicting population changes over time. Using the model, they discovered that a too aggressive policy would result in a complete collapse of the population if a decreased birthrate or other perturbation reduced the stocks below the minimum level. The U.N. and speeding ticket groups both sought decisions that would optimize some variable. Optimization is the most common goal for student questions. The illumination group was interested in a design that would improve their immediate working environment (their classroom was a windowless basement room which made lighting a major concern for these cave-dwelling students).

Objectives

In a vital democracy, a primary goal of schooling should be the development of thoughtful, informed, and active citizens. Mathematics is an indispensable tool for reaching this goal. With mathematics, we can ask and answer important social, scientific, and political questions and analyze the claims that policy makers present to us. Mathematical modeling skills and attitudes will help our students become more questioning and more curious and, therefore, less passive and less gullible receivers of knowledge. Toward that end, students completing a modeling course or sequence should be 1) informed, skeptical consumers of publicly disseminated mathematical and statistical claims and 2) able to take an issue of interest and use mathematics to better understand it, to discover new facts about it, or to generate new questions for thinking about the matter. Most often, public discourse about the importance of mathematics education emphasizes preparing students for a career and the significance of this preparation for the future of our economy. Students who have taken my modeling course have returned to tell me how much it has helped them in their work in science and industry. Both the political and personal contributions of these studies are essential and should receive equal attention.

An additional objective of the modeling curriculum is to re-establish the value of liberal arts learning. Each discipline represents a different approach to looking at and understanding our world. Each discipline has its strengths and weaknesses, and each should be taught as a growing field of knowledge. The teaching of the disciplines is sometimes derided as outdated and intellectually confining. These objections have grown because the ideas and importance of the subjects themselves seem to have been lost in the process of creating curricula that focus on technical skills. Consider, as evidence, history classes that are dominated by memorization rather than primary source research and critical analysis. Similarly pointless are geometry classes in which students regularly prove meaningless claims about measurements in arbitrary diagrams.[1] These experiences fail to focus on the goals and purposes of their disciplines.

Each discipline should be learned in ways such that they become a tool for discussing and making sense of real world questions. Mathematical modeling is that part of mathematics which focuses on these objectives. Students will only understand that we study mathematics for this reason if using math to understand their questions is a regular focus of their education.

On the first day of classes, I give my students a goals sheet (**I1.1**[2]). This sheet poses the essential intellectual questions that will guide the course and lists the habits and skills that they should seek to develop in order to grow as mathematicians and learners. The teaching of each goal is discussed in the next chapter. I refer to the goals sheet regularly in determining the design of my classes. It serves as my list of axioms and any activity that does not follow logically from them (i.e., help encourage a habit or provide insight into an essential question) is avoided. Specific reference to traditional mathematical content areas is omitted not because new mathematics topics will not be addressed, but because they are the means and not the ends of the course. These ends could be met through the study of a range of mathematics subjects. The content areas that we explore, with additions based on students' interests and questions, are outlined in the content strands below. Topics are chosen in order to deepen students' understandings of central mathematical ideas and to broaden their views of what mathematics is and what mathematicians do.

Whether one teaches a course on modeling or incorporates modeling into some larger curriculum, I urge you to help your students pose and grapple with their own questions. They should spend considerable time using math to resolve problems for which mathematics is neither the initial source of their motivation nor the end result. When we teach mathematics as an intellectual lens through which non-mathematical questions can be examined, then we are

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1 Proof is central to mathematics, but makes sense only when real conjectures are involved and new theorems are being built upon the foundations established by earlier proofs.

2 Handouts are indicated by a letter which indicates the chapter and numbers which identify the activity and their associated documents. The complete list of items can be found in Appendix A. Handouts are provided as separate files so that teachers can modify them according to their needs. Teachers have permission to use these materials solely within their own classrooms. No version of these documents may be distributed without the author's permission.

providing our students with an education that will serve them and our society throughout their lives.

The Pedagogy of Modeling: Philosophy and Strategies

Affect and Persistence

The habits of problem posing, creating representations, explaining connections, and testing and checking are central to the development of interesting new mathematics and applications. Students need to see these habits as worthwhile activities. Real world applications often involve many variables, incomplete information, multiple methods of solution, and answers that vary according to the assumptions and simplifications made and approach taken. Encounters with such settings dispel students' notion that the trademark of mathematics is the exactness and uniqueness of results. Rather, recognition of underlying structure and abstraction become dominant features of the discipline. We must help students become comfortable with uncertainty while striving for clarity in their descriptions and analyses. They must accept that creativity and clear communication are part of active learning and discovery. Lastly, they must be curious and willing to take risks. Successful students in traditional math courses are rewarded for speed and technical accuracy. A different type of confidence is required when they begin posing problems with no immediately clear method of solution and no guarantee that a solution can be found.

The first homework assignment (**I1.2**) asks students to reflect on the goals listed in the goals sheet (**I1.1**) and the challenges that the goals pose for them in the coming year[3]. Because the content and approach of this course are often a radical departure for students, it is crucial that they be aware of, and ultimately value, the course's objectives. My classes regularly discuss the meaning and purpose of the goals in order to develop this support. I do not assume that enrollment in a class equals approval of, or commitment to, the course goals, nor can such approval be forced. It must be negotiated, encouraged, justified, and inspired.

The three quotes below are excerpts from students' responses to the first option from the goals reflection assignment (**I1.2**). These responses reveal several common reactions: fear, optimism, and skepticism. Many other students write about which goals they like because they already feel competent in those areas (e.g., working independently or writing clearly). These essays are the start of a dialogue. I add positive and encouraging comments (samples in italics) throughout each by agreeing with their feelings or observations. I will offer help to a student citing a particular weakness (e.g., calculator use) or additional opportunities for an individual to let me know if I am providing the support that they need to reach these goals. Few students

[3] Although this curriculum can be integrated into other courses as a strand, I will describe it as a single course because the sequence and connections should be attended to in either case.

choose the second essay option, which is a more difficult question. For either question, students sometimes give answers without providing much explanation. For example, favorite or feared goals may be noted without reason for the reaction. A student might suggest that they keep a diary in order to demonstrate their progress toward meeting the goals without detailing how the record would demonstrate improvement. As with all work, I point out the need to elaborate on comments and to justify their claims.

> I noticed three things that I have never had to do for a math course before. These goals may take me a while to get used to and/or get good at. They include defining math problems and asking math questions as they relate to society and nature, knowing and identifying which math tools to use when solving a problem and writing narrations of exploration and problem-solving efforts.... In all of the math courses I have taken in the past the teacher has taught us exactly what we needed to know to go on to the next year's courses, nothing more, nothing less. We were never allowed to stray too far off topic, thus we didn't pose too many questions of our own. The problems we did do came from the book and they didn't seem to relate to nature or society. Due to my past experiences I do not have much practice in the realm of asking math questions or coming up with problems from the real world. As a result of this I am a little apprehensive of this course but hope that I will learn quickly. *Be patient with the process. You are right in noting wholesale changes in expectations. Give yourself time to adjust.*
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> <u>Articulate your thoughts and discoveries</u> is a goal that is not 100% to my liking, yet strikes my curiosity. Rarely before have I ever written narrations of my discoveries and problem-solving techniques. I believe that if I am capable of explaining my actions in writing, then I would definitely be able to understand what I was doing since writing and explaining could be considered as my weaknesses. *You are right in noting that it is impossible to write clearly until you fully grasp an idea yourself, but the writing itself can facilitate thinking. Your writing here reveals your thoughtfulness.*
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> When I see the heading "Enjoy Mathematics", I am immediately inclined to think, "Yeah, right." I have never enjoyed math, but that may be because I have never had a chance to really apply it to real life. For example, in Algebra classes and such, the closest we ever got to applications were word problems, which are pretty much laid out [for you]. *As the year proceeds, please let me know if you continue to find this goal implausible and we will search for an intersection between your interests and our studies.*
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> Excerpts from Student Responses to the Goals Reflection assignment (**I1.2**).

Two months into the course, I provide written evaluation on each student's progress. I note how they have fared with the goals that concerned them. Students often make the most progress in those areas that they were aware of enough to write about in the first place. If not, I offer advice on how to start improving.

The first two attitudes and habits listed on the goals sheet are of paramount importance to a student's success and continually challenge me each time I design a new activity. The goals of this course are for students to engage in lengthy, complex projects, to take risks and grapple with deep ideas, and to be sufficiently satisfied with the experiences that they want to face such difficulties again in their lives. Realizing these goals requires persistence and a tolerance for frustration. Students (and adults) can become paralyzed, depressed, or antagonistic when faced with open-ended, incompletely defined tasks. The activities described here are designed to gradually raise the level and length of the challenges, but no amount of easing in can eliminate the symptoms just cited. (I had a colleague who brought puzzles to class every Wednesday. On entering the room, his student's would hold up their fingers in a cross to ward away the "devil.") It is essential to make clear for the students what they will be facing, how they may feel, why you are asking them to go through the frustrations they will meet, and how you are going to support them throughout the process. After the goals sheet (**I1.1**) is discussed on the first day of class, I distribute copies of the following quote:

> It may be that when we no longer know what to do, we have come to our real work, and that when we no longer know which way to go, we have begun our real journey. The mind that is not baffled is not employed. The impeded stream is the one that sings. - Wendell Berry

I ask for interpretations and then tell them that my job is not to be the Grand Explainer (although I explain ideas regularly), but to be their Baffle. My job is to create interesting situations with impediments, the overcoming of which will lead to significant progress in their understanding. My job is also to provide a framework for approaching challenging tasks and to coach them when they are stuck.

Many students interpret the above quote as a message that they are being left to their own devices. On a daily basis, I remind the class, or a recalcitrant individual, that they are not expected to sink or swim on their own. They are encouraged (and sometimes shanghaied) to come to me to ask questions about a topic or problem. I try to help them develop the ability to self-assess when they are really stuck and need to brainstorm with me. I will approach students who cannot tell or who think asking for help is wrong. It is sometimes difficult distinguishing for the students and myself where the line between increasing independence begins and avoiding floundering ends.

Students are much more open to meeting unsettling expectations if they, in turn, have some control over how the course is structured. Students are understandably fearful of failure. I always negotiate due dates for major assignments so that the students are confident they will have the time to do the assignment well. Students can also receive an extension when they have worked hard on a problem and want to continue their investigation provided the extension is requested in advance[4]. They are allowed to redo efforts that were not successful to see if they can perfect a skill or discover a solution to an unsolved effort[5]. One year, a student proposed a touch-base session halfway through the ten days that they had to work on a problem set. They suggested that we devote a class period to individual work and easy access to me to ask clarifying questions. Both sections voted to implement this idea. I was not thrilled with this use of class time, but it provided structure that several students needed. They were having trouble spreading out their work over the entire ten days. Having this plan in place helped them start productively when they first received the problem set and avoid last second efforts. During the class, I would check each student's progress and gain unambiguous information about how they were using their time. The touch-base sessions also reduced the stress some students felt asking for individual help outside of class time (they felt that the need for extra help reflected poorly on them). Writing clear, open-ended questions can require careful wording to maintain both clarity and open-endedness. The meetings provided a good opportunity for me to refine my instructions on particular problems. This experience reinforced my belief that when students make academic decisions, the payback for trusting their judgment will be considerable.

Another way students are given control of the course is through the attention paid their ideas, questions, and interests. This attention is central to meeting the goals that students be investigative, make connections, and identify topics for exploration. Students only embrace these goals if the sought-after behaviors get the response they deserve, namely class discussions and assignments move in the direction of their thoughts. Valued behaviors are repeated because class time is devoted to them. One year, as a result of student questions, a class spent two weeks discussing Flatland and higher dimensions and the other studied game theory and the Prisoner's Dilemma. Although the thought of having students choose course content may disturb some, nothing is a clearer indication that students are developing particular mathematics interests and actually want to do math. One consequence of this approach, which I warn them about ahead of time, is that I do not hand out syllabi. I explain that either the syllabus would change daily or I would have to ignore those comments that did not head in the pre-planned direction. Students

[4] This rule is in place to discourage last second efforts and to place a minimal requirement that they realize at least 24 hours in advance when they have not used their time well. It is a early step that they need to take monitoring the progress of lengthy projects.

[5] Students do not abuse the extension and redo policies, because they do not want to fall behind or face too many overlapping assignments. In fact, I wish they would take advantage of the redo option much more than they tend

who see content-laden syllabi quickly shut down their own initiative and mathematical problem-posing.[6]

A teacher must also be willing to repeatedly and patiently explain the reasons for the design of a course and choice of topics. Students should be able to ask, “Why are we doing this?” and get a thought out answer that is rooted in the intrinsic value of the task (i.e., “it will help you later” should not be the typical response). One girl once accused me of inflicting “Mr. Abrams’ Math” on them and claimed that what we had studied (number properties and levels of infinity) would be useless both in later life and as preparation for Calculus. I explained the role infinity and infinite processes play in Calculus, pointed out the importance of understanding the behaviors of numbers we too often take for granted, and made a plea for appreciating the aesthetic and paradoxical aspects of the ideas we had been exploring. She was not convinced, but she was mollified and some frustration had been vented. Students regularly note the openness of the dialogue and flexibility of the course as significant in making them more comfortable with the demands made upon them.

The most important determinant of the students’ affect will be the teacher’s. I am always excited about math and its applications and share that enthusiasm unashamedly with them (liking math does, of course, make one suspect in this country). I get excited by their ideas, work on their problems, bring them articles and books on topics of interest, and get happily perplexed by difficult problems in front of them. When I am struck, I do my best to work further on the problem and bring back my results or latest dead-ends until I have made some progress.

The development of persistence in both individuals and a class requires careful planning. It is important that students have some early successes, but these need not always be immediate or complete. Challenges of greater length and difficulty should be introduced gradually and students need to have most of the skills and understandings required to solve a problem (with just one or two gaps which may be the sought after goal of the activity). Students are not only frustrated by the difficulty of the work. Modeling introduces some students for the first time to the subjective nature of mathematics used in context. The possibility of multiple interpretations in a field heralded for its precision and lack of bias can be unsettling. Lastly, work on projects follows rhythms (of effort, insight, and types of tasks) that are not present in shorter exercises and to which it takes time to adapt. Both teacher and student need to be patient with these changes.

The main goals of the course are demanding and non-negotiable. I acknowledge students’ feelings and support them as I can, but when they ask for lectures and chapter tests I decline. It is

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to. It takes a particularly motivated student to be willing to rework a “completed” assignment. Drafts (built-in redos) are part of several larger assignments.

6 Several essays and books cited in the bibliography extend this discussion of control in the classroom.

alien to them to be told that struggling and being perplexed should be fun and is an efficient way to learn. Humor is indispensable as kids discover that whining and threatening will not help them. During a difficult investigation one day, a boy asked where I lived so he could burn down my house. The class laughed and I did, too, but I did not give him my address! Another student repeatedly told me I was a terrible teacher. I responded that she was right in her case, because I was failing to convince her that mathematics, as I understood it, was worth her time. I also promised to keep trying to persuade her.

Most students initially consider a mathematical modeling course weird and not like the math with which they have always succeeded. As a school year progresses, only 2 or 3 students maintain an edge to their comments and, by the end of the year, all of the early detractors comment that they now see math in a new light and have learned a great deal. One student sheepishly "admitted" deciphering bar codes on her mail. Others feel more versatile and able to appreciate and work with abstraction. All develop a critical eye and greater patience with big projects. One student who was never hostile but often felt defeated made considerable progress in the sophistication of her thinking and clarity of her writing. She was even able to work in a good-natured way on a proof for several weeks in May without ever finishing it or giving up. Accepting the lack of a guarantee that her efforts would pay off in the most gratifying and clear-cut manner was an indication of her deeper appreciation for the process of doing mathematics.

> Can the bubbles on a straw in seltzer accumulate such that the straw accelerates upward and injures the beverage drinker?
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> What portion of a religious group should actively proselytize in order to maximize membership in the religion?
>
> How can one eat a Reese's Peanut Butter cup in more than one bite and still have the same ratio of chocolate to peanut butter in each mouthful?

While modeling requires intellectual rigor and risk-taking, it also encourages the development of a personal mathematical aesthetic and, as evidenced by the three project questions above, of whimsy. An enjoyment of intellectual playfulness can create a strong intrinsic interest in a project. Similarly, teenagers respond positively to discussions with any moral or political controversy in them. Modeling problems can expose and inform controversial questions[7]. My students have demonstrated time and again their enthusiasm for using math to answer their own questions, for the role that creativity and curiosity play in complex tasks, and

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[7] If a class is not getting caught up in an animated conversation occasionally, it tells me that either the students do not yet feel safe expressing themselves (in which case we have a discussion about good discussions (see Developing Good Class Discussion Habits, page 25)) or I have not yet found topics of interest to the class.

for the chance to have opinions about subjective and important problems. Their originality fills me with appreciation and wonder every year.

Just as students have their own aesthetics, so too do teachers. We all have our favorite topics and activities that get us really excited. It would be just as foolhardy to suppress this built-in motivation for us as it is to limit our students. I have avoided creating a formal textbook precisely because of the constraining nature of both traditional, and many reform, curricula. Both variety and flexibility are essential. On the other hand, you can never please everyone. One of my most talented students, who loved pure mathematics, reflected on his modeling experiences in a written evaluation this way: "I learned that you shouldn't go using something beautiful like math in real life situations. I learned there was a seamy underbelly to math."

Exploration and the Use of Class Time

> "I don't care whether math is fun or not. What I care about is that it is meaningful. When you have meaning, you get the fun for free." – Doug Jones

Placing problem posing and problem solving at the center of a course forces many changes in the use of resources. The primary resources are time and literature. Since the doing of mathematics is the content of the course and doing math is difficult, unpredictable, time-consuming work, substantial quantities of class time must be given over to student thinking and investigation. Discussions often proceed deliberately because a student's question will often be followed by time for the class to reflect and respond. Everyone in a class needs time to critically analyze a question. Likewise, the usual eager, hand-waving volunteer should be encouraged to check the validity of their first instincts. If no one has an answer, students will usually pair off to discuss the problem and come up with an approach. If different answers arise, students present their reasoning and seek consensus. I usually play moderator in these discussions without offering an opinion. In my refusal to play expert, students learn that there is no quick route for finding the 'right' answer. They must refine their arguments and listen to their classmates' reasoning. In the face of disagreements, the students seek to be more articulate, to present more persuasive justifications, or to be less convinced by their own way of thinking.

I have not found shortcuts to teaching the skills of probing, interpreting, playing with, and communicating ideas and do not believe they arise spontaneously from exposure to routine mathematical exercises. Thus, I have had to develop the patience to allow my students to be confused and uncertain and to work their way, as a group, to new understandings. The philosophy behind these methods is essentially a constructivist one. Constructivism claims that real understanding, which must be complexly connected with prior learning, can only develop through the active discovering, explaining, and testing of relationships by students themselves[8].

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[8] See Blais (1988) and Brooks (1993) for further discussions of constructivism.

It is difficult to make a timetable for discoveries. The “aha!” moment is surprising precisely because we do not know when to expect it. Real learning is messy and unpredictable. While lecturing (or “directed instruction” as I have seen it euphemistically called) leads to predictable schedules, it fails to produce the conditions that will help students meet the goals listed for this course. If students are excited enough by an idea to want to spend more time on it, to delve deeply, then much more will be learned than if I push them on to the next topic. The final modeling projects (see the Projects chapter) have grown from eight days to five weeks in response to student demand for more time to get farther with the process. If we seek creativity and wisdom from our students, we need to be flexible ourselves.

Content Choices

> "I can think of two good criteria ... for deciding [what to teach]: whether the knowledge gives a sense of delight and whether it bestows the gift of intellectual travel beyond the information given, in the sense of containing within it the basis of generalization…. [We should focus on] depth and continuity in our teaching rather than coverage."[9]

The Mathematical Modeling course is composed of several content modules. The topics have been chosen because they form a core of activities that are fruitful for teaching the skills of modeling. While all high school and college math topics prove useful solving one modeling problem or another, the goal is not to teach a catalogue of already known important models.

The first unit of the course introduces the process of creating, evaluating, and learning from mathematical models. This modeling cycle provides a framework that connects all of the different habits and skills which students practice throughout the course. The second and third units explore the properties of different number types and function families. They teach students that real world problems involve both data and information about the behavior of the entities under study. The statistics unit can be studied fruitfully at any time after functions are well understood. I try to schedule it in order to take advantage of links with students’ science curriculum and research projects.[10] Open-ended modeling projects, described in the final chapter, are the culmination of the course. Topics such as probability, combinatorics, sequences and series, fractal geometry and chaos, and graph theory are used to enhance students’ understanding of the breadth of mathematics and the range of modeling questions that one might ask.[11] For

9 Bruner, Jerome. *On Learning Mathematics*. Mathematics Teacher. December 1960, vol. 53. 610-619.

10 Descriptions of the statistics, probability, and simulation unit will be added to this document in the future.

11 These topics are never all explored in a single year. Many mathematics topics receive attention each year as a result of the questions which students pose. For example, a project group that wanted to design a profitable

example, questions about how likely a phenomenon is or how many different ways an event might occur are promoted by the first two topics. For all topics, an effort is made to start at "an interesting middle"[12] with questions that introduce the students to the subtlety, fun, surprise, or utility of the subject.

It is impossible to create a coherent course without first listing clear and connected course goals (such as **I1.1**). Once such goals exist, they should be foremost in all planning. Lessons should be designed with attention to which goals will be served by the activity. If no goals or essential questions are furthered, then maybe the lesson should be skipped. Alternatively, the need for a new goal may have been identified.

The following additional questions should be addressed when weighing how class time should be apportioned between various topics and activities:

> Is the topic new? If it is a topic that students have had in prior courses, why is it being taught again? If the students did not learn it the first time, then we are obliged to teach it in a different context and fashion. We are similarly obliged if we are teaching a topic in order to extend what they know about it.
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> Does the topic have immediate applicability and interest? Classes should focus on thinking and technical skills that are not merely prerequisites for some possible future mathematical experience. Often, new technical content can serve, in parallel, as the vehicle for highlighting a modeling skill.

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casino game ended up studying matrix operations and Markov chains because those ideas were important for their work.

12 I heard this expression at a talk years ago and cannot remember whose creation it was. It is a good rule of thumb for avoiding introductions that dwell too much on preparatory minutiae (which can be returned to once kids are engrossed in a set of ideas). In general, the overarching themes of a topic should be addressed early on. For example, most high school and college texts treat combinatorics as a set of counting techniques (permutations, combinations, etc.) and students are stymied if a problem does not match one of the presented formulas. While combinatorics is about counting, the main questions should be: How might I count this set? How can I tell if I over counted or undercounted? How can I adjust my answers if I know I am off? How can I turn this problem into one that I do know how to count (mappings and isomorphisms are important mathematical ideas which can be explored in this broader context)? With these questions, problems that are not related to factorials (e.g., the number of distinct pentominoes) may still be investigated.

Is the result of a given activity of consequence? Have students found out something that is useful to know or have they merely completed an exercise?[13]

Choose topics and activities that make the habits of thought accessible. For example, teaching some pure math questions may be quite helpful in developing modeling habits if the connections are made explicit (see the discussion on 10^{50}-10^{20} in Technology and Magnitude in the Number chapter). Similarly, pure math problem posing skills can be central to good modeling (visit Making Mathematics at www2.edc.org/makingmath for materials for teaching mathematics research).

Each year that I have taught Mathematical Modeling, I have presented fewer and fewer topics and made each surviving activity correspondingly longer. Survey courses strike me as nearly useless, because students' recollections of all the glimpses offered by such a course quickly disappear. A focus on a few main ideas leads to long-term mastery and understanding. The more our students and we practice techniques in a variety of rich contexts, the more proficient we will be with them (i.e. able to use them in expert ways outside of rote exercise situations). On the other hand, it is helpful for kids to know that math is a vast subject. I show them the list of sub-topics into which Mathematical Abstracts (the periodical guide to published mathematics papers) divides the field. This list dispels any notion that math is simply a linear sequence from arithmetic to Calculus. They also read interviews with, and biographical essays about, applied mathematicians from different mathematics sub-fields (see the bibliography for two examples).

Students taking mathematical modeling do develop many new technical skills and understandings. However, the emphasis is always on learning how to apply their knowledge, from throughout their mathematics careers, to problems in ways that were not obvious at the outset. It is this criterion that distinguishes exercises from problems. Textbook exercises are usually associated with a particular recently learned skill that is being practiced (if you just read a section on trigonometric ratios, you can bet they will be helpful in solving the questions that follow). In contrast, problems do not come labeled "I am just like the question solved on page 263." It is this absence of instruction about which skills are appropriate, and the need to figure out for oneself what to do, that makes them problems. Since modeling begins with a real world question, it is applied problem solving. Content should be chosen, in part, for the degree to which it lends itself to valuable problem-solving experiences.

[13] For example, can a real question or data from another discipline be used in place of an artificial setting or data set?

Resources: Information and Research

Textbooks frequently determine the content, sequence, and methods of the course with which they are associated. The fatal flaw for most texts is that, in their attempt to be both curriculum and teacher, they necessarily have to explain everything to try to avoid confusion. This incessant explaining preempts conjecturing and discourages the asking of interesting questions. Even for inquiry-based texts, the physical presence of a bound sequence of investigations sends a message that student ideas need to follow a particular path.[14] It can still be helpful to have a traditional text handy as a reference for students. One text that students have used for Mathematical Modeling is Advanced Mathematics by Richard Brown. This book was chosen because it is an up-to-date Precalculus text with varied sets of exercises, chapters on a wide variety of topics, and few substantial links between the units forcing any particular order of study. The book is sometimes used for standard homework assignments, but more importantly it serves as one resource for students to use when they need to review a topic or find out about a skill that they need to solve a problem.

Almost every assignment the students submit throughout the year is completed “open-book.” Thus, they need to learn how to find information in their text, in the library, and on-line as appropriate. This use of resources moves their focus from memorization to the identification of what they do not know. Then they must figure out what parts of what they need to know might be available in the literature and how to obtain that information. Researching needed information includes a range of tasks that require persistence and creativity. As much as my students would like it, the Internet is not the sole repository of information. They need to be able to interview people, search through library stacks, track down leads in bibliographies, accumulate potential keywords for subject indices and library card catalogues (real or virtual), and read through technical literature looking for accessible portions.

Each student needs access to a graphing calculator. I also frequently use a computer lab, but several of the computer activities can be translated to the calculator. Students benefit from access to spreadsheet, statistical, geometry, graphing, and symbolic manipulation (e.g., Mathematica) software. Technology is used because, in the exploration of functions and analysis of data, it changes the nature and number of questions that students can ask. As with any tools, students need to learn the benefits and limitations of its use. For example, students who attempt to “number crunch” an answer to every difficult combinatorics problem, rather than patiently analyze the structure of the situation, usually fail to reach answers that can be generalized. They

[14] Some elementary curricula, such as the excellent Investigations in Number, Data, and Space or the FOSS science modules, avoid putting texts in student hands, but should still do more to help teachers structure activities that follow up on fruitful, but unanticipated, student ideas and questions.

are likely to miss the insight that a pencil-and-paper route might have provided. On the other hand, there are questions that do not readily yield to analytic methods, but are profitably explored using computer simulations. The main goal is for students to take advantage of technology's ability to facilitate rote work and expedite deeper conjecturing about patterns and structure in mathematics.

Mathematical Modeling is inherently interdisciplinary. Modeling endeavors require the modeler to combine their mathematical expertise with knowledge from one or more other fields. Frequently students have to become "instant experts" in their area of interest in order to craft an effective mathematical representation. Thus, as they are developing as mathematicians, they are also learning how to be independent learners in all fields. The motivation to do this work arises from the fact that they are answering questions that they themselves have posed. The diversity of interdisciplinary connections made is inspiring. Student problems are drawn from the realms of physics, political science, sports, education, public policy, economics, religion, biology, architecture, and even gastronomy. One modeling group asked, "What is the most cost effective time to purchase a new computer system?" The optimal time for a purchase depended on the useful lifetime of a computer system. The group's attempt to define obsolescence included six separate approaches. One definition was based on the availability of compatible software and another on the Internal Revenue Service's allowable period for depreciation. The snow mound group (page 2) researched the physics of the absorption and reflection of the sun's energy as well as the thermal conductivity of ice. These students needed access to a decent academic or public library, newspapers and magazines (which are stacked around my classroom), and a variety of textbooks, encyclopedia, and almanacs on different subjects.

The most important resource that the students have is each other. They stimulate each other's interest in the discipline and in the challenge of problem solving. They are encouraged to work on homework and many of the major assignments in groups and class activities are usually cooperative in design. Students report a considerable amount of spontaneous peer tutoring during study times.

Assessment

The goals of assessment are for the student and the teacher to be informed about the degree to which the student is meeting the objectives of the course, the ways in which the student accomplishes their work and thinks about the material, and the skills and understandings that need further practice, refinement, or reformulation. The Mathematical Modeling curriculum follows the assessment philosophy detailed in the NCTM Assessment Standards. Assessment is a daily task that is undertaken by both the teacher and the students.

Informal assessment occurs in several ways. Homework is, in general, not collected. Students are instructed to check the validity of their efforts using methods suitable to the work. For assignments that involve questions in the text, they are to check all answers that appear in the back of the book and rework incorrect exercises. There are many students who require continual monitoring to get them to take advantage of these answers. For skills studied in the text, students are required to come to class with specific questions, knowing what difficulties they have or have not encountered. Students are often paired off to try to solve problems about which either may still have questions. As students confer, I circulate looking at a few questions from their homework that I choose ahead of time to study on each paper. For each problem, I check their notes, look to see how detailed their work is, seek evidence of second efforts, and identify which methods they employed and what reasoning they used. As I scan their work, I scratch quick notes on each thus producing a qualitative record of their nightly homework. Similarly, throughout the year, I record comments on noteworthy observations, questions, techniques, behaviors, or efforts exhibited by the students. I try to note something about each of them every week.

When a particularly interesting or fundamental question is asked during class discussions, I want to know how everyone would tackle the problem or whether everyone has the basic understandings I am assuming. In these cases, I ask everyone to write a response rather than continue the discussion. As they write, I quickly walk around the room reading their responses, sometimes only half complete, and can determine efficiently where the class stands with an idea. This technique is much more efficient than calling on a single individual for a response and allows me to note problems or particularly creative attempts that might otherwise be skipped. It also keeps all students involved. All of the above routines provide me with information about the students, but I also strive to let the students know what I am seeing, to ask about their thinking on a particular point, and to congratulate them on something they may have done in class.

There is considerable variety in the design of class activities and major assignments. Class time can be devoted to discussion, computer labs, investigations of a question, time for groups to work together on a project, or peer evaluations. Homework is a mixture of one-day readings or text exercises with long-term problem sets, research papers, group projects, and experiments[15]. Timed tests play a very small role in the assessment activities of this course. Typically, there have been two traditional quests[16] and one essay-based midterm. There was also one timed,

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15 Students frequently need to juggle small nightly assignments with ongoing long-term projects. I regularly remind them of their over-lapping obligations and help them learn how to organize their time.

16 This is how I refer to most collected efforts, thus avoiding the questions “Is this is quiz or a test?” or “How much does this count?” I also explain that quests are a search to see what the students can or cannot do at a given moment. The word has a more romantic and less educational feel.

cooperative, computer-based quest. Timed tests are used when there are basic skills or facts that are worth memorizing. Since most endeavors are long-term and open-book, memorization is not emphasized. Tests are also avoided because interesting problem-solving is not encouraged by an emphasis on speed. In administering tests, one is obliged to teach good test-taking skills. The main test-taking approach everyone learns is to do the problems that are easiest first and skip the ones that will require too much thought. Timed-tests require quick thinking and quick surrender and undermine the goals of persistence and challenge. We tell students to check, but we rarely give them enough time on tests to do so. Furthermore, creative exploration often arises from repeatedly playing with, mulling over, leaving, and returning to a problem. Our minds are wonderful parallel processors and will often think about a problem while we're doing something else and then notify us when it has made some progress. Few traditionally taught students have had such "aha!" experiences because of the absence of long-term assignments.

The variety and length of assignments can be stressful in their newness, but also contribute to helping the students accept the goals of the course. Timed-tests are by their nature stressful as is the search for some particular right answer or approach to a problem that one may think should be thought of differently. In contrast, open-ended problems may permit multiple means of solution, multiple interpretations and answers, or even multiple new questions. It is this latter habit of creatively modifying and extending problems so that a single problem becomes an area of investigation that I cite as a pinnacle of achievement for the students. Discovering that a question that interests them will probably intrigue some of their classmates makes the students more willing to invent and work together as a community of mathematicians. The stress associated with long open-ended problems is minimized because my feedback and evaluation is based on any good thinking that is in evidence. Students are rewarded for what they do well, not what they fail to do or cannot yet do. The students vary in the skills that are improving most, and must learn to use their best tools while they develop the others. For example, one student might rely on graphic arguments to make a point while a classmate may prefer more abstract, symbolic manipulations to reason about the same situation. As we all try to do in our jobs, they get to play to their strengths while they eliminate their weaknesses.

Lengthy tasks are also emphasized because how a student learns is fundamentally connected to what they learn. Unless, one engages in complicated processes, the organization and thinking skills required will not evolve. Additionally, skills that are only learned in one context will not readily be transferred to other applicable settings. When students are repeatedly required to apply multiple skills in a variety of settings then the scope and utility of those skills will become apparent and students will seek to apply them in order to understand future situations. This philosophy claims that not only do complex thinking habits fail to emerge when we focus on teaching simpler (frequently called 'basic') skills, but that those basic skills themselves are not fully understood[17]. A focus on projects also alters students' view of mathematics and what mathematicians do. If activities are not rich, then they assume with good cause that the subject is not either.

Careful reading and clear writing are at the heart of all assignments. Students are forced to think about and understand an idea with greater rigor when they have to communicate that understanding to another person, particularly a reader who is not an expert with the problem. Classmates regularly read and give feedback on each other's writings. Unlike a teacher who may struggle to make sense of an awkward passage and will look for the germ of truth amidst haphazard reasoning, peers will simply write "I do not understand this paragraph. You need to define this term and stay on topic."[18] I also give in-depth feedback on the thinking and style of their writing. At the beginning of the year there are always objections such as "This isn't English class! Why are we writing? Why are you commenting on grammar?" I explain that effective communication is not limited to one discipline and is probably the single most important skill one could bring to the working world. They quickly learn that they must present conjectures and evidence in a clear and organized fashion if they want to convince anyone of the accuracy or beauty of their ideas.

I do not know a high school or college teacher who has not had cheating in their classes. This course is not immune to that problem. Students can cheat on projects by plagiarizing solutions or by getting help from classmates, friends, or tutors for those problems sets that have expressly required an individual effort. It is impossible to restrict the opportunity to cheat if you want to see students grapple with problems over a period of days. A couple of years ago, students did inform me that there had been cheating on the first problem set. I believe that instances of cheating provide a crucial opportunity to explore with students what school is about and what their responsibilities are to themselves and to the creation of a community to which we would all want to belong. I told the classes that cheating had occurred (I did not know by whom) and that the course could not continue until we had decided how to avoid this problem in the future. The classes then talked about the problem and I moderated the discussion and took notes. I made very few comments. The following day I distributed copies of my notes (**I2.1**), which were quietly read, and then the discussions resumed. This process continued for three days before the students felt there was sufficient class consensus that teachers were reasonably powerless to stop cheating, that cheating was potentially poisonous to the community and their intellectual progress, and that they were more willing to actively monitor the problem.

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17 Note that this view is in exact opposition to the view espoused by Saxon Publishers that both basic and sophisticated thinking emerges from the repeated practice of algorithms.

18 I find that peer comments have been quite helpful and remarkably "teacherly" in tone.

I do not know how much cheating took place thereafter. I believe it was, at least, greatly diminished. I believe this experience was a turning point as the students realized both the power they were given at their school but also the obligations that accompanied that influence. Several students told me they used to allow classmates to cheat off them at their former schools and that they no longer felt this was acceptable behavior. Another informed me that he now felt he would have the courage to tell classmates to stop cheating if he saw them and to even turn them in if they ignored him. Some expressed cynicism about what the "real world" was like and whether it was possible to be any better than what they saw waiting for them. Later discussions frequently touched on questions of democracy and citizenship.[19]

Coaching

"If a student seems to need help, seem to help them." - Henry Pollak.

The first teaching task of the year is to assure that students understand and adopt the work habits they will need to stay organized and prepared for their projects. Because the course does not follow a text, note-taking during discussions is imperative to having a record of the ideas and skills discovered by the class. Students who lack thorough notes find themselves ill equipped for the problem sets. For the first several weeks, I randomly choose one student at the end of class to submit their notes, which I copy and return. This process not only encourages the students to take good notes, it also creates a file of daily notes, which absent students can readily consult. I tend to let this routine fade out later in the year because I want the students to recognize the utility of taking notes and to do so for that reason, but the advantages of keeping them focused during class and of building a notes file may outweigh this objective.

Problem solving involves being stuck. If a task does not puzzle us at all, then it is not a problem. It is merely an exercise. When the students ask me a question, my usual response is a question. My first question usually asks if they can clearly state what they are seeking to determine or if they can figure out why they are stuck at a particular point in a process. Often students stop in the middle of a task but do not try to characterize what has occurred that has

[19] The class discussions frequently returned to grades and pressure. I make sure that students know that I do not curve. Curving has two negative consequences. When students do poorly, but get higher passing grades, it tells them there is no absolute standard for progress. When they all do well, but get lower grades due to the comparison, it forces an artificial scarcity of rewards for excellence. My term grades are also not strict averages. Re-do's are usually permitted and progress with a previously unsuccessful task is noted. Grades should reflect what a student can do by the end of a course, not just at the time of their first crack at a skill. Taking this notion further, perhaps successful students in an Advanced Algebra course should have their Algebra grade retroactively changed to an A!

stopped them in their efforts. Encouraging them to identify the cause of their ‘stuckness’ (e.g., I have too many variables, I don’t know how steep this driveway is) is frequently all that they need in order to focus on, and resolve, their difficulty. As I ask them these questions, I also make explicit what I am doing and why. I am asking them the questions that I would ask myself if I were in their position. I encourage them to set up their own internal dialogue in which they continually ask themselves “What do I know?”, “What do I need to know?”, and “What techniques do I have for bridging the gap?”

Sometimes the needed level of internal dialogue is strikingly simple. On numerous occasions, students have come for extra help because they were stuck with a multi-step problem. After showing me the first step, they would freeze. I ask them “OK, what do you do next?” They do one more step and, once more, stop. I repeat my question and they do one more step until the problem is solved. They thought I was being helpful, yet I point out that all I did was prompt them to continue and that uncertainty should not be allowed to lead to paralysis.

Students will become more comfortable “talking to themselves”, if they see the teacher doing likewise. When students ask a question which I cannot answer, I will often begin working on it in front of the class. I will outline my intuitions about what the answer might be and how I think a solution might be reached. I will try methods and, if they fail, backtrack and start new pathways. All the while, I will share with them the questions I am asking myself each time I face a next step (how do I know it is time to factor, or graph an equation, or). Teachers must be willing to give students these types of apprenticeship experiences. We expect our scientists, athletes, or artists to learn by studying expert practitioners of their crafts. Similarly, math students need to see what it looks like to do real mathematics. Students who continually try to “trip up” the teacher with questions that he or she cannot answer are not troublemakers; they are a font of opportunities for modeling problem solving.

A common feature of difficult problems is that the initial information is ambiguous or incomplete. Problem-solvers often make assumptions about the setting that they do not realize they are making. These assumptions may be unwarranted or simply may alter the nature of the solution. In either case, I try to help the student think about the information or properties that they assumed applied. This metacognition is crucial but not simple. The background that we bring to a problem is not always explicit in our thinking. Asking a student to state what they know and to explain how they know it can be a first step to clarifying their reasoning. What many students do not realize is that “thinking about something” is an active process. Thinking should involve questioning intuitions, monitoring reasoning, checking definitions and computations, modifying methods and testing cases, planning strategies, comparing results with expectations, or communicating a clear description of your work. For example, information can be represented in different forms. Thinking about a formula might be aided by trying to put the relationship into words. A set of specific numbers might be more revealingly studied through a graph. Thinking requires repeated forays into new representations and transformations of an idea until a useful perspective is discovered. I try to make these activities apparent to the students when they exhibit them and to emphasize that what they just did is what we mean by “think about it!”

I do explain many ideas to students. A goal of this course is for students to learn how to discover new information. The goal is not for them to discover every new fact or skill they need to know. When I explain an idea, I try to model how I might figure out the relationships being studied or I compare the different ways I think about an idea as it appears in different contexts. Sometimes a student asks a question which they lack the mathematical tools to analyze. In such a case, coaching should involve sparing them a fruitless search. I might direct them toward literature that will enlighten them or tell them they have asked a deep and nifty question and teach them about a pretty approach to the problem.

An advantage of question asking over telling is that I am in a better position to monitor my students’ learning when they are doing the talking and explaining. Furthermore, by not telling them when I think they are right or wrong, they are forced to take checking seriously and to talk to each other and make a case for their work. The class becomes a mathematical community instead of a collection of student-teacher dialogues. Furthermore, if I am the ultimate source of ‘Truth’, then my students will be ill-equipped once the course ends to continue using mathematics on their own or with their peers.[20]

<u>Understanding and Supporting Group Efforts</u>

One of the course objectives is effective participation in group endeavors. Students often encounter difficulties working together, so I have added peer and self-assessments of group process and behavior. Prior to the ladder project (see Exponential Ladders in the Numbers chapter), the class writes responses to and then discusses three questions. Answers to *“Why do we have groups working together? What are the goals of cooperative efforts?”* include: 1) There are tasks that cannot be accomplished by an individual in a reasonable amount of time. 2)

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[20] The effect of these methods comes across clearly in the following course evaluation: I am more confident about my math now. I'm willing to say what I think is right and back it up with proof probably because I find it easier to have reasons that back up my thinking. One thing I've learned this year is to be more skeptical of things and to figure them out myself rather than take someone else's word for it. One thing that helped me be more confident was the fact that no matter if we were right or wrong you would ask "why" and we had better have a reason. Another thing was that you would never tell us the answer. No matter if the test was over or the problem set handed in, you would still only give us hints so that we could figure it out ourselves. You always believed that we could figure it out ourselves so we did.

Collaboration, when effective, is capable of producing outcomes that are more interesting, creative, and thorough (checking each other's assumptions and computations) than any of the group's members could produce on their own. 3) Group work requires the development of listening and communication skills that have importance outside of formally structured group endeavors. 4) It facilitates the sharing of ideas and the gaining of different perspectives. 5) It brings together the different backgrounds, and therefore different strengths, of the group members leading to a greater appreciation for each others' talents. 6) It is more fun than working individually (remember that enjoyment was a course goal). One optimistic individual suggested that groups were preferable "because it is better to fail with someone than alone."

The second question, *"What issues arise when groups work on a task? What factors affect how well or poorly a group functions?"* has elicited the following concerns: 1) How much time each person invests in the project. Is the distribution of labor fair? 2) Is there agreement on the group's purpose? Are there common standards and an equal commitment to the task? (If not, effort may be unequal). 3) Do the individuals have different work habits or approaches? Are they organized and cooperative? 4) Do they have similar or different problem-solving styles? Complimentary or conflicting ideas and philosophies? 5) Is there good listening, good will, or competitiveness? Is communication honest but constructive? Is there clear expression of opinions and ideas? 6) What roles do people play in groups? What structures are possible? 7) Are the group members friends or even friendly? 8) Is there enthusiasm for the project? Is the task understood? 9) Are individuals focused and patient?

In one class, one student wrote that groups work well when there is a recognized leader delegating responsibilities. Another said groups fail when one person tries to tell everyone what to do. It was clear that expectations could play a large role in the success of a group. This disagreement highlighted the need to explicitly explore students' conceptions of effective approaches to group work. Neither view is wrong, but the combination of views left unexplored would assure a poor experience. The class in which this difference arose brainstormed clues, signals, and feedback that would help to establish or change the relationships group members had so that a structure that served the members and the project was chosen. When students discuss troubles that can arise, they do a better job avoiding the pitfalls.

Lastly, students respond to *"What can you do if a group is not working well?"* They suggest: 1) Discussing the problems as a group. 2) Talking with an individual whose behavior is of concern and trying to find out why they are behaving a certain way. 3) Taking a break. 4) Getting a moderator. 5) Encouraging each other. 6) Taking the initiative to lead and set an example rather than nag. 7) Talking with the teacher. I encourage them to use me as a last resort although I may notice a problem and intercede with an observation. I have never honored a request to break up a group, but I also make new groups for each project.

It is important for all of us to be able to provide and receive constructive feedback. Students need to feel that it is safe to do so and to be comfortable that the process is geared toward their personal and academic growth. After the ladder project, the students provide constructive peer feedback. I ask each of them to write their reflections on how their group worked during the previous week. They answer the following questions: *How did your group work together (an "objective" description)? What worked well? What could have worked better? How could it have worked better?* I promise I will read these comments only to assure that the feedback is clear and responsible and then pass them on to the student's partner(s). Each group member receives the same grade for a project. The feedback will not be graded or influence the grades of the individuals in the group. This promise can be kept by organizing the feedback after returning the projects, however immediacy does add to the impact of the process. I do give feedback to the feedback givers as well. I have had students complain to me during a project and then be completely glowing in their feedback. I have experimented asking students to assign a percentage of the effort to themselves and their partner. Remarkably, there is usually considerable agreement between partners about how much each contributed to the final product. Although equal credit is awarded to all partners of a project, I do talk with those students who self-report less than satisfactory effort.

Although it is not always recognized at the beginning, students quickly realize that an overriding issue affecting the success of a group project is the availability of time to work as a group. Depending on where group members live and on other life circumstances, time may not be available for meetings outside of class. Groups can sometimes be formed with these constraints in mind. In general, a significant amount of class time may needed to provide work time for the groups. This time will be most productive if, at the end of each class work session, groups are required to assign and record individual homework tasks to be completed prior to their next meeting.

Group work exposes students intimately to the ideas and approaches of others. In small group settings, students are more likely to be really listening and less likely to be distracted than during whole group discussions. Quieter students who hesitate to participate in class become more involved and everyone is more willing to experiment, take risks, make mistakes, and analyze their reasoning.

There is no one best way to create groups. They can be formed based on which research question each student wants to explore. Students can be allowed input into the formation of groups ("give me a list with your top ten choices"). Groups can be formed randomly (picked out of a hat). Avoid attempts at tracking students into groups that are all supposedly of the same or different ability. There are too many important skills (problem posing, representation, writing, organization, proof, etc.) that contribute to research for there to be one tidy ranking within a class. Research efforts are worthwhile, in part, because they make apparent to the students that mathematics is not primarily a test of computational accuracy and that they each have a range of skills that they can bring to meaningful mathematics explorations. A well-balanced group will give each student a chance to be the expert at some task. A group that uniformly lacks a skill (e.g., using technology) will be forced to work, as a group, to gain some mastery in that area.

Developing Good Class Discussion Habits

The above activity for developing a whole class understanding of the reasons for, and ways to improve, small group efforts can serve as a model for discussion of many aspects of classroom life. For example, the questions can be applied to considerations for preparing and giving an oral presentation. While small groups are an efficient way to have many students talking and thinking at once, they do have the disadvantage that each group is unaware of the ideas and methods being explored by the rest of their classmates. Whole class discussions provide a means for involving everyone with the same questions and for calling attention to important skills and concepts. These discussions can be enhanced by a preparatory exploration of discussions themselves.

The following prompts can yield a good list of guidelines for how to be involved in a discussion:

- Why do we have class discussions?
- What factors contribute to a good discussion?
- What are the students’ roles during a discussion?
- What is the teacher’s role?

Some student responses include: Discussions bring out different points of view, give you a chance to rethink opinions, and force you to make your ideas clear enough for yourself so that you can communicate them to others. Discussions succeed when there is active participation by a variety of people (and quieter kids strive to test out their voice occasionally); speakers are audible; people know what they are talking about; comments are sincere (people stick to claims that they believe). When they are not speaking, students should be active listeners (make eye contact, nod, or otherwise communicate physically that they are being attentive). Everyone should be courteous, respect other people’s opinions, and be open to other people’s ideas. The teacher (or an appointed student) is a good facilitator when she calls on folks fairly, reminds people of the standards for involvement when needed, poses thought-provoking questions, makes sure no one dominates, and clarifies comments (Kathleen Ennis, personal communication).

Note that not all of the ideas raised by students may be the best choice for a given class. For example, students often suggest hand raising, but a fluid discussion may emerge when people try to enter into discussions in a less formal, conversational fashion. Even with hand raising, it can be the responsibility of the last speaker to call on the next one so that everyone grows in his or her awareness of who needs to be brought into the discussion. A class is more likely to develop into a mathematics community when the teacher is neither the primary controller of discourse nor the main creator and transmitter of knowledge, but the facilitator for learning. During discussions, avoid as much as possible being the sole arbiter of what answers are correct and what reasoning seems valid. When students look to the teacher for the final word on a question, they are quitting on their own responsibility to struggle with the evidence for themselves.

There are several additional steps that you can take to facilitate group discussions. One is to emphasize for students that they are engaged in a *discussion* and not a *debate*. Many students see discussions as a chance to show off, to impress the teacher, or to prove their superiority. Reminder the class that discussion goals include learning from others and appreciating the complexity of ideas being considered. The physical layout of the class can also help or hinder discussions. Chairs should be as circularly oriented as possible, so that everyone can talk to the entire class.

Discussion skills are another area for coaching students over their emotional barriers to progress. Many teenagers develop the protective habit of starting a question with “I have a dumb question...” or “I know this is wrong, but....” It is important to privately point out this habit and encourage students to drop these tag comments. They need to hear that questioning is not a sign of weakness. Good questioning is crucial to an interesting class. They also need to help their listeners take their comments seriously by not deprecating themselves at the start.

Acknowledgments

My courses have grown organically over the years subject to countless collegial and curricular influences. I have tried to identify sources and give credit where due. I believe the true creative challenge of curriculum development for a teacher is how they pull together all of their borrowed ideas into an intellectual experience they can best teach. I am particularly indebted to my two high school teachers and first bosses, Stewart Galanor and Antonia Stone. Their imprints are still clear in my work. Henry Pollak is responsible for my, and many others’, interest in Mathematical Modeling. Some activities are of my own devising, but many are modifications of ideas from the journals that I scour and from my colleagues. Foremost among my colleagues/mentors have been Karen Bryant, Kathy Ennis, William Luzader, Gerry Murphy, Manya Raman, and Janet Youngholm. I am grateful to all of them for the insights that they have shared. Lastly, I want to thank all of my students over the years who have shared their enthusiasm, ideas, opinions, fears, triumphs, and lives.

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Brown, Richard. Advanced Mathematics. Houghton Mifflin, Boston, 1992. *Both of these texts cover a wide range of mathematics topics that can be used to supplement this material to provide practice in more traditional technical skills. The first book is closer in spirit to this work. Both are broad enough to serve as reference texts.*

Hugh Burkhardt "Mathematical Modeling in the Curriculum" in Applications and Modeling in Learning and Teaching Mathematics Edited by W Blum, et al. Ellsi Horwood Limited, John Wiley and Sons, Chichester, England, 1989.

Doty, Lynne. "Real Mathematics for Non-Majors." PRIMUS, Vol. 5 No. 3, September 1995, pp. 205-217. *A fine example of a curriculum that was planned with intellectual habits (and not technical math skills) as the primary goals. Doty discusses how she uses very accessible problems from graph theory to introduce the skills of understanding definitions, making conjectures, and proving theorems.*

Blais, Donald. "Constructivism - A Theoretical Revolution for Algebra." Mathematics Teacher (November 1988): pp. 624 - 631. Brooks, Jacqueline Grennon and Martin G. Brooks. In Search of Understanding: The Case for Constructivist Classrooms. Association for Supervision and Curriculum Development, Alexandria, VA, 1993.

Mathematician Essays

Albers, Don. "Making Connections: A Profile of Fan Chung." Math Horizons (September 1995): pp. 14-18. *Chung's work at Bell Labs centered on discrete mathematics. She presents a graph theory problem that students can understand and play with.*

Albers, Don and G.L. Alexanderson, eds. Mathematical People: Profiles and Interviews. Birkhäuser, Boston, 1985. *The interview with Persi Diaconis discusses his work in statistics,*

approach to finding new problems, work debunking psychics, and teen years as a magician. Diaconis is a colorful figure.

Appendix B - Handouts*

- **I1.1** - Goals Sheet.
- **I1.2** - Goals writing assignment.
- **I2.1** - Cheating Discussion Notes. 3 Not on disk? Not a blackline master.

* Documents are numbered within each unit and sub-unit (N3.2 is the second document of the third sub-unit of the Numbers in Context unit).

An outline of the main goals for Mathematical Modeling

Essential questions to explore

- What is a mathematical model?
- What are some approaches to creating and modifying models of real world situations? What are the different ways of representing a real world situation? What are the benefits of each representation or mathematical tool?
- What knowledge can be derived from these models? How do we gauge the reliability of that information? In what ways can there be more than one right model or answer?

Habits and attitudes to be developed and extended

Enjoy mathematics.

- What you learn this year will not benefit you unless you look to apply it on your own after this course. Wanting to continue studying and using mathematics throughout your life is more important than any given skill that you learn now.

Embrace challenging tasks. This objective requires in turn that you:

- Are persistent in your efforts.
- Expect to be investigative and to ask questions.

Pose problems.

- As with all disciplines, mathematics provides one way of understanding our world. For it to become a useful tool for you, you need to define problems and ask questions that you have about nature and society.

Simplify and generalize.

- Seek to greatly reduce the complexity of the settings that you explore. Look for the broadest possible application of the results which you obtain.
- Increase the realism and complexity of your methods gradually, proceeding only when each stage has been mined for all possible insights.

Create mathematical representations that help you understand your world.

- Beginning with data from or reasonable assumptions about a phenomenon, make informed choices about which of your mathematical skills could be applied to yield new insights about that setting.
- Provide justification for the decisions that you make. Be able to test your models and offer a judgment as to their reasonableness.

Make connections.

- Actively search for links both between different mathematical ideas that you have studied and between mathematics and other realms (science, art, current events, etc.).
- Seek to pose new problems for study, to ask “What if…?” questions that add a new twist to a situation.

Analyze the models and statistical claims which you and others make.

- Be able to interpret what you read and are told and to assess the validity of the reasoning that led to the stated conclusions.
- Seek to explain *why* the results that you reach are the case.

Check your reasoning and solutions.

- Only accept conclusions if you can verify or estimate the validity of an answer.

Articulate your thoughts and discoveries.

- Write narrations of your explorations and problem-solving efforts.
- Be able to clearly communicate your ideas to an appropriate audience (e.g. a classmate, newspaper, etc.).

Work cooperatively on explorations and projects.

- Strive to share ideas and listen to others to learn from them.
- Set standards for effective group work.

Work independently.

- Seek to identify topics of particular interest for independent exploration.
- Assess your efforts (Is this work logical? Creative? To the point? As good as I can do?).

Use technology.

- Know how and when to use calculators and computer tools. Understand the limitations and power of each.

All of the above have these overarching objectives: That you come to view the ideas and information presented to you during your life from a mathematical perspective (just as historical, aesthetic, and other perspectives guide your understanding). That you can use mathematics to generate new ideas and information.

Assignment

Re-read the goals sheet for the course. Respond to one of the following questions about the goals listed in the habits and attitudes section (1 page, more or less).

a) Choose some goals to which you have a stronger than average response. Are there goals that you particularly like or dislike? Are there goals that you think may be easier or more difficult for you? Please explain why they are more or less appealing or more or less challenging for you.

OR

b) Some of these goals are more general and some more tangible than others. Pick a few goals and discuss how would you demonstrate to someone, or measure for yourself, your progress toward those objectives.

Notes From Two Junior Classes At A Public Magnet School .

Comments and Questions from Friday's Discussions

Comments

Several suggestions were made on ways to reduce cheating (different structure, easier questions on timed tests, etc.).

Some felt the rules were not clear or the circumstances of problem sets made it difficult to treat the assignments as solitary challenges. Others felt the rules were clear.

Ultimately, everyone agreed that the teacher can't stop students from cheating.

People talked about the distrust, frustration, and intimidation that was growing within the class. The cheating had strained some of the class cohesion.

Some said they were embarrassed by the cheating and resentful that they were struggling while others were getting better grades.

The prospect of students confronting each other or reporting cheating was discussed. The KGB was mentioned. Others saw these actions in the positive light of taking group responsibility for what takes place at the academy. Some worried that classmates who cheat would not respond to the concerns of their peers anyway.

It was pointed out that most efforts in the class can be redone, so cheating for a grade seemed odd. It was also noted that Mr. Abrams can be asked questions if someone is stuck.

Several people felt that take-home work will always be cheated on.

Procrastination made the thought required by the problem sets impossible and that caused some to cheat.

Many of the academy activities encourage group work and it is hard to break out of that habit. The problem sets should allow group work.

None of the math assessments involve simpler problems/work. There should be more of a balance.

Several people spoke emotionally about their desire to keep the problems sets because of the thinking it allowed them and because of the excitement of succeeding with a challenge.

Some did not want to lose the honesty and freedom that they felt they had at the academy and not at their former school. Each person's character and approach to life was called upon. Others talked about the friendships at the academy (how they are affected by the cheating and how they might be used to eliminate the cheating).

Problem sets are better than timed tests because the point of math class is to develop your thinking.

Some would be angry if they lost the problem sets even though they weren't getting good grades on them.

Questions

Why are people cheating? What is the purpose? What is really being lost and who is really being cheated?

How can the temptation to cheat be reduced/removed?

What are each person's responsibilities?

Is there such a thing as responsibility for the behavior of the community? How do we respond to these broad responsibilities if we accept them?

What are we going to do with the function problem sets (which everyone agrees are not all legitimate)?

What are we going to do with the current statistics problem set?

How should the course proceed?

Comments, Questions, and Proposals from Monday's Discussions

Comments

Redo option is only an option for people who have extra time. There is not enough time to rework efforts and strive for excellence. You can spend hours thinking about something and no thoughts come.

Friends should help each other be better people and not be resentful if they are held accountable for their behavior. "Finking" is a good thing for the finkee.

"Finking" will lead to problems in student relationships. Cheaters will ultimately have their behavior catch up with them. Calculus next year will be tougher for confused cheaters.

Colleges expel cheaters. We need clear rules regarding breaches of the honor code. Elite institutions are, and should be, strict about academic dishonesty.

Students should deal with cheaters one on one.

Clarifications: The school does not compute class rank. Math problem sets, quests, etc. are not curved. Re-worked effort that demonstrates new understanding is accepted as just that: evidence that you have met an objective of the course. Grades are not strictly averaged, they reflect your success in meeting the goals of the course (earlier difficulties become unimportant).

At old schools cheating was rampant and received little punishment (detention). People were not used to taking it seriously, but are more upset by it now.

The discussions are helping people realize that they shouldn't cheat.

Too much attention is being paid to negative consequences rather than positive reasons for learning (not cheating). We need to deal with transgressions on a case-by-case basis.

We should not change the problem set structure because people need to learn how to do long projects, to budget their time, and to be persistent.

The problem sets ask us to reveal our thought process and not just the answer.

Mr. Abrams talked about Haverford and their honor code and the responsibilities, privileges, and pride that exist there.

If people want this school to work then they have to take the responsibility to act. If they do not act, they won't become the people they want to be.

"I have big dreams for what we can do as a community."

Why are people cheating? What is the purpose? What is really being lost and who is really being cheated?

- People are stressed about comments during parent-teacher conferences about being kicked out at the semester mark.
- People who cheat don't have sufficient motivation.
- Pressure. We have been chosen. We have responsibility to our parents who expect learning and getting the grades. Grades can blur your thinking.
- Getting in to a good college.
- People are used to getting good grades and this course has been particularly difficult.
- What we are losing is people's integrity. People who do the work are still under a cloud of suspicion.
- Some who have tried are getting discouraged (not feeling smart even though they are). They are tired of being impeded streams.
- Intellectual frustration. Want to know how something is solved, to see how to get a result. People need the ability to accept mistakes.
- "The Big Empty 'A' syndrome: good grades but no understanding."
- Some people let others cheat off of them in order to be helpful or show off, but they aren't really helping them. Grades are focused on too much among the students.
- Grades can be a good motivator. The pressure can be helpful and the grades can get you where you want to go in life.
- Grades are totally unimportant and yet parents ignore the teachers' written comments.

What are we going to do with the function problem sets (which everyone agrees are not all legitimate)? What are we going to do with the current statistics problem set?

- Function problem set should still be counted because many people worked hard and deserve the credit. We should wince and move on and accept that some will get credit they don't deserve.
- People should have the integrity to turn themselves in still.

The classes should be polled anonymously to allow self-reporting of cheating.

Questions

What is accomplished by telling on someone?

How can the temptation to cheat be reduced/removed? Can the impact of grades be changed? What role do grades play? What would we do if there were no grades?

What are each person's responsibilities?

Is there such a thing as responsibility for the behavior of the community? How do we respond to these broad responsibilities if we accept them?

How should the course proceed?

Proposals

Problem sets should have a check-in class early on when Josh will answer questions and note initial progress (pro: provides some structure for long assignments. con: takes up a class period).

The writing of a school honor code should get underway.

A student handbook detailing the schools expectations and standards should be produced.

Each member of the academy is responsible for their own behavior, for upholding the integrity of the endeavors of the school. Helping someone cheat is cheating. If you see a student or students cheating, you (or a group of students) should tell them to stop.

Students who are caught cheating should be told to report themselves to the faculty.

Students who fail to report themselves to the faculty should be reported by those who witnessed the violation of the school's honor code.

Transgressions of the honor code should be individually reviewed by a student-faculty committee.

Requests

More frequent town meetings for group discussions.