Mathematical Modeling: Teaching the Open-ended Application of Mathematics

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An Introduction To The Modeling Cycle

The many skills of modeling are taught in the numbers, functions, and statistics units. The process that brings all of these skills together is known as the mathematical modeling cycle. This cycle is studied and practiced in all of its richness prior to and during the final modeling projects. It is crucial, however, that students have scaffolding that provides context for the individual skills as they are learned. They need to see how the pieces of the course fit together. This unit introduces them to the modeling cycle.

At the start of the course, I define mathematical modeling for the students. Now it is time to define models themselves. Students answer the following questions:

"What, in a non-mathematical setting, is a model?" Responses include an object that is bigger or smaller (a scale model) or a representation that is simpler than the original. Models are a substitute for the thing that they model.

"Why would we use models instead of the real thing?" A model may be easier to study. It is less effort to test a model of a bridge than it is to build the bridge. In this same example, we see that a model can also be a cheaper and faster alternative. Because a model is a simplification, it may be more revealing than the real problem. The countless variables affecting human health obscure some health hazards that are more readily detected in animal "models" in controlled laboratory situations. This example also highlights that a model may also be safer to use (at least for humans). Students note that models are frequently tangible; they can be physically manipulated and studied. A model can be controlled and adjusted (it is a *working* model), while the real setting may be inaccessible. For example, an astrophysicist may explore models of stellar evolution by varying the constituent elements or arrangement of particles in a proto-star.

"What are the objectives of modeling?" Prediction (e.g., weather forecasting), design, the testing of possibilities in order to make a decision, or the development of a deeper understanding of a phenomenon are all common goals. Models allow us to ask "What if…?" questions, to imaginatively explore situations that might not actually exist.

Just as a scale model, such as a toy boat, represents a more complicated object, capturing some of its characteristics (appearance, proportions) while simplifying or neglecting others (size, same materials, function), so too a mathematical model represents a situation symbolically, graphically, and/or numerically retaining the aspects that are essential for study and putting aside details of lesser importance. Because of this simplification and because mathematical objects, such as equations, have known properties and behaviors, studying a mathematical model of a real-world situation can provide students with insights that are hidden during a nonmathematical study of the same situation.

The process of creating a model and using it is an involved one. The class reads, and then I explain, the modeling cycle diagram (**P1.1**). The first, and for students most unfamiliar, stage is the creation of the mathematical representation (box #1 and the top arrow). The act of creating a model forces the modeler to think deeply about the setting. Translating an imprecise, complex, multivariate real-world situation into a simpler, more clearly defined mathematical structure such as a function or a system of rules for a simulation yields several benefits. For the first step in this process, modelers identify a list of variables. As they do so, they discover what they really know about their problem and what information they need to determine through library research or experimentation. In choosing a particular type of representation, they must think about the connections between and among variables, decide which relationships and structures are most important to capture mathematically, and pick the mathematical realm that offers the best possibilities for expressing all these features.

Once a mathematical model exists, the technical skills of traditional school mathematics come into play. Algebraic expressions are simplified and factored, equations are solved and graphed, and geometric figures are studied for special properties (the Manipulation arrow in **P1.1**). As new mathematical knowledge is derived, it is translated back into the real-world context (the Interpretation arrow). For example, if the mathematics forced a domain restriction on a variable, this information might imply a not-yet noticed real-world limit as well. All new conclusions must be compared to what is known and tested for believability before even provisional acceptance is granted.

Modelers must begin as simply as possible when they set out to create a manageable first model. For example, the group studying the melting of snow mounds ignored the three-dimensional world and created a model of a vertical plane of snow. Reducing the dimensions simplified the geometry and algebra of their model, but left a physical setting that seemed sufficiently realistic to provide meaningful first answers to their question. Similarly, exponential functions are frequently presented as models of population growth and with good reason. Births are generally proportional to the size of a population. However, human populations rarely display such simple behavior for long periods of time. The exponential function derives from a first trip around the modeling cycle. Further cycles might incorporate other variables, such as changes in health and nutrition, immigration, wars, or the age distribution of a populace.

Once a first model is fully understood, additional variables or connections between variables can be incorporated into an increasingly complex representation of the setting. This habit of initially excluding seemingly important aspects of a problem is counter to most students' instincts. The necessity of creating a preliminary representation, which does not fully solve a problem, needs to be illustrated, emphasized, and practiced repeatedly.

Political Parties and Street Vendors

Activity: *Are Two Parties Too Few? An Introduction to the Modeling Cycle.* This lesson introduces students to the stages in the development and analysis of mathematical models. They learn about and practice the steps in the modeling cycle by exploring, critiquing, modifying, and testing a model of competing political parties in a democracy. They then consider the implications of their findings for the design of political systems.

Technical Skills: Number line understanding, simple fractions.

Modeling Skills: Identifying variables, clarifying assumptions and simplifications, choosing appropriate mathematical representations, translating mathematical structures into real-world conclusions, testing alternative models, extending an existent model.

Interdisciplinary Connection: Politics in America is often discussed in terms of a political spectrum on which most voters and candidates can be placed. History, constitutional requirements, and legislation have combined to create a system in which two political parties vie for the support of the electorate. The dogma in our country is that two parties provide voters with the real choices needed in a democracy. Many democracies have more than two effective political parties. Is there a significant difference between the election rules in these countries and is there reason to believe that one approach to designing a democracy might be preferable to another? The benefits that we ascribe to our political and economic systems can and should be tested.

Audience: Students with at least a basic familiarity with the election process. Younger students can engage in a parallel exploration with the Vying Vendors simulation and software (see Teaching the Modeling Cycle to Middle School Students on page 15).

Materials: Several pennies (or other markers) for each pair of students, a copy of the political party or vendor sheet (**P1.2** or **P1.3**) for each student, disks with Design Your Own Democracy (available for both Macintosh and PC computers) or Vying Vendors (available for Macintosh computers only), and lots of blackboard space (or supplement a small blackboard with paper on an easel or an overhead – the students may generate lengthy lists). Software may also be pre-loaded on your computers.

Length: At least two hours of discussions and one session of computer explorations.

Students are introduced to the modeling cycle as they investigate and critique a mathematically simple model about a topic that is familiar. The model demonstrates how different laws or voter behaviors may affect the changing political positions that parties adopt over time as they maneuver for advantage (Hotelling 1929, Pollak 1980). What follows is an abridged transcript of a class with additional commentary interspersed:

Class begins with a discussion of the steps of the modeling cycle. *Today I will be presenting a model I've been studying. The development of a model requires a setting of interest and a particular question. The setting for today's model is a sequence of elections between competing political parties and my question has to do with the range of political positions adopted by the parties. The stages of the modeling cycle are: 1) Identifying variables and needed data; 2) formulating a model (mathematizing the situation); 3) deriving mathematical results from the model; 4) interpreting those results in light of the real-world setting; and 5) returning to step 1 until the model seems to produce realistic behavior. We will start in the middle of the modeling cycle. I will start by describing my model in its simplest form and give you a chance to play with and understand it. When you encounter a model, seek to understand its purpose first and only later attempt to evaluate it in any critical fashion. This model is about political parties, the political positions they stake out, and what happens to those positions over time. The model is a simulation like SimCity. A simulation requires the creation of rules that describe the behaviors of the entities you are studying. Once the simulation is carried out, you can observe the consequences of those interacting rules. The entities within our simulation are voters and parties.*

The voters in this model exist along a spectrum from left to right and the parties also occupy positions along this political spectrum. Because we are the United States, we will start with two parties. I hand out two pennies (other markers are OK) and a number line with circles numbered from 1 to 11 (**P1.2**) to each student. *Each penny represents a party. Place your pennies down on two random spots. A party at 3 is taking that moderately left position between the extremes. Read the rules on the left regarding the voters and work with your partner to determine how many votes each of your two parties will receive.* Students discuss the rules and try to apply them to their situation. I check to see if each group has correctly determined the vote total for their starting positions. *Now read the rules on the right, which describe how the parties change position. Run your simulation according to those rules, keep alternating turns, and see what happens over time.* (See Figure 1 below for a sample run on a smaller board).

Soon after beginning the simulation, groups encounter the problems presented by adjacent and overlapping markers. For example, if marker *A* were at position 8 and marker *B* were at position 9, then *A* would receive 8 votes and *B* would receive 3 from the voters at positions 9, 10, and 11, which were closer to it. If it were party *B*'s turn, then a move to position 8 would garner a total of 5½ votes, since each marker would be equally close to all voters and all eleven votes would be split. A student's question, "Two markers can't be on the same number, can they?" immediately takes the class into the thick of modeling decisions. *That's an important question, which is not covered by the original rules. What would the real-world interpretation of that overlap be? What would a prohibition of marker overlap mean in the political setting? I'll let you decide whether to allow that possibility or not, but justify your decision based on which choice seems most realistic, or try both and discover how each influences the model's behavior.* Very different conclusions emerge from the two options. For example, if students prohibit the overlap because they believe that two parties would never have common platforms, then one marker can pin another one in an unfavorable position.[1] The class discovers how significant their modeling choices will be since such different conclusions result from different rules. They also learn that every aspect of their model should represent some characteristic of the real-world situation.

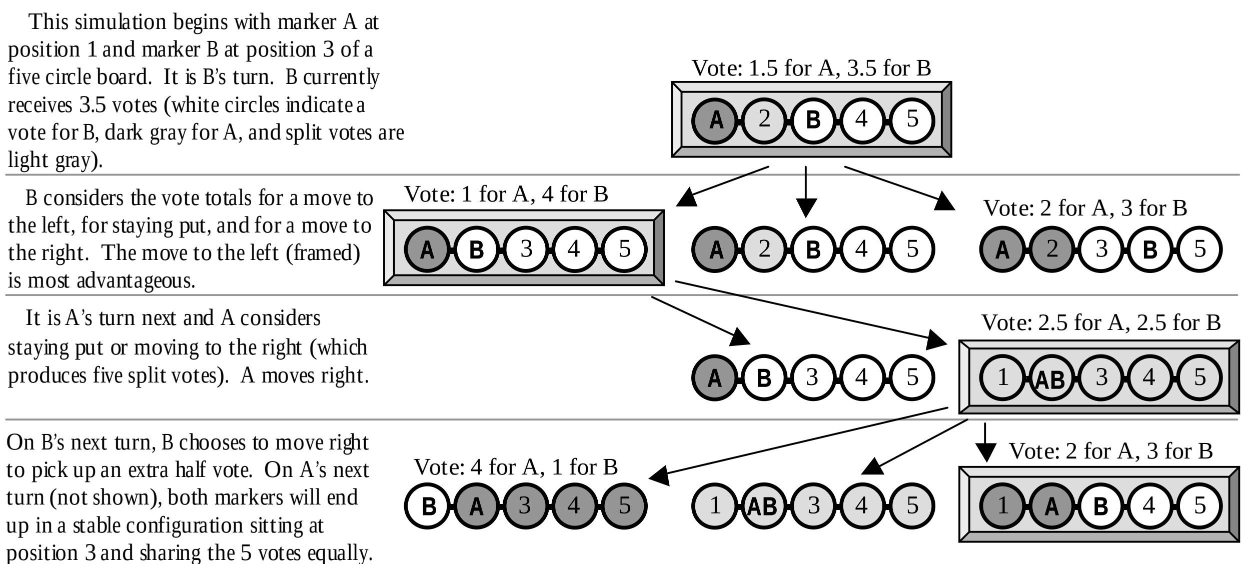

Figure 1. The Political Parties Simulation.

While the students carry out their simulations, I go around checking to make sure that they are always choosing optimal moves for their parties. Once the simulations have reached a stable configuration, I seek comments and observations. Students point out that the pennies freeze on

[1] The possibility that one party might be able to hem in another seems to me to be more of an artifact of the discrete nature of the voter spectrum than a valid representation of our political life. Certainly, candidates can leapfrog each other (consider some of the class views espoused by Patrick Buchanan, the initially-Republican candidate for President in the 2000 election, which in many ways were to the left of Democratic platforms).

the central circle (circle #6). Don has stopped with parties at positions 4 and 6, which is balanced at 5.5 votes apiece, but is not, according to his classmates, stable. *Now the purpose of modeling is to try to understand something about the situation we're modeling so could someone interpret what this model tells us about our political system (again taking the model at face value)?* Laura takes issue with the model, noting the even distribution of voters. She comments that there are probably more voters toward the center of the political spectrum. *That is a good observation about the true nature of the voter distribution, but let's stay focused on what we can learn from this simple model first. Critiques are not allowed yet!* David points out that all politicians end up as moderates. Laura disagrees saying that if one party were at 0, the other would want to be at 1. I agree with both claims. Ben says they are still being forced to converge somewhere. Mita clarifies the matter by pointing out that they will end up at the center, that Laura is talking about a moment in time and David's claim is about what will occur eventually. *The class has just succeeded in working through the modeling cycle from box 2 to box 4. That is, you have carried out the simulation (the manipulation arrow) to generate a mathematical result and then interpreted that result as a real-world prediction. Now you are at the Analysis stage: Do you see this result as accurate, as what is happening in America?* Yumi responds that the Democrats and Republicans want to distinguish themselves; to do what they said they would do. Matt feels the result does show what we see – the politicians say what will get more votes. Pam says the real political parties do not shift as much as those in the model do. Laura, citing a statistics paper on spotted owls that she wrote, sees convergence in politics. Her paper discussed the compromise between the environmentalist position of no logging and corporate desires to preserve logging. Ben agrees and points out specific Republican politicians who downplay their conservative views on abortion and Democrats who distance themselves from traditionally liberal stances. *The cynical among us have company. A student last year brought in the following cartoon* (see above).

If, as this model suggests, two parties do not offer voters distinct choices, is there any benefit to having a two-party system versus a one-party system? Caroline says that there is and notes that under this model one party could sit wherever they want uninfluenced by public opinion. She puts a penny down on two and observes that they get all of the votes regardless of their position. The two-party system eliminates extreme positions and, in some sense, better reflects the average values of the electorate.

My interest in the model came from the observation that in a two-party system you would see this convergence and from my feeling that voters are not given much choice in elections. I was interested if this apparent homogeneity had to do with how our democracy was designed and whether a two-party system could ever provide us with choice. So my original question was actually more specific than I told you at first. It was "how many parties do we need under this model to get distinct platforms?" As a voter, I want not only to have a choice of parties, but to have a choice that is not too far from my own views. Note that this goal does not assure voting for a winner, just for a party with which you agree. Now, with your partner, pick a number of parties greater than two, get more coins, keep track somehow of which party's turn it is, and see what happens over time. Be careful, systematic. After 20 minutes of work, students report their findings for different cases. Groups studying four party systems discover several, reasonably similar, stable configurations involving two pairs of parties. The pairs cover a larger span of the electorate and are close to most voters. I ask for vote totals each time. *Why do we get different arrangements?* Pat says the final state depends on the starting position. Carol asked whether it is better to maximize your votes or to maximize your net vote (i.e., maximize your lead or minimize your deficit). In a multi-party race, these two strategies might produce different final results. Some groups discover that three parties cling to the center of the spectrum but never reach a stable arrangement. Instead, the parties follow a periodic dance around each other (returning to the same position every six steps). Some students begin to explore the periods for different numbers of parties. Others want to experiment with the effect that the initial position of the parties has on the ultimate outcomes. *So according to this model you need 4 political parties to give voters real choice (distinct options but not necessarily nearby choices).*

I want you to take four minutes to pick up with what Laura started before. That is, I have presented a very, very simple model of a very rich phenomenon. Identify the areas in which I simplified the phenomenon, that is the areas in which reality and my model don't agree. Think about what I described as the model and the most recent elections through which you have lived. Make a list. I scout around and notice on the paper of a student who likes to be funny: "Politicians aren't pennies." He also has some serious items. I tease him about his first entry and when I call on him later, he offers, “Voters aren’t circles!” Sample critiques[2] and questions follow:

The model assumes that…

- Everyone votes (100% participation) and that parties always split contentious voters.
- Voters won’t be turned off by constant shifts in position and become disaffected.
- Parties are willing to readily change their stand. However, a classmate replies that this simulation spans a lengthy period. Each turn is an election and we should compare the simulation changes to those that have occurred over the past 20 or so elections in the United States.
- There is an equal distribution of voters and an equal spacing between discrete positions. It is difficult to know what the spectrum represents, but the distances may not be equal.
- People vote for a political position or a party platform and not for a specific candidate (voters seem less party aligned than they used to be). They might vote on experience, cuteness of candidate, character attacks, etc. Does an independent candidate function differently than does a party? What would the difference be?
- People know exactly what the candidate stands for. The candidates know exactly what the voters think. Everyone is omniscient (polling perfected!).
- A party’s candidates and issues can be represented by one position on a single spectrum. Candidates have multiple issues (e.g., economic, foreign, environmental, civil liberties issues). A “pro-choice” fiscal conservative and a “pro-life” New Dealer would be difficult to position along the model’s simple linear political map (as would a libertarian, etc.).
- Money and the media are not factors in the process.
- The model omits primaries and multiple candidates from one party.
- Does it extend well or match European multi-party parliamentary situations? Are the 4 party results a good match to these situations?
- There is no historical record, so having power does not lead to future advantage. Use a big coin for the incumbent and award ties to that coin giving the advantage to the party in power.
- Parties do not occupy a single spot. They spread out over a broader sweep of opinions. We could use dollar bills to cover more spots.

[2] Countless other critiques are possible. Some include: There are only 11 voters. The voters are unaffected by peers, polls, or lawn signs. Voters are static – real voters concerns can change with the times. Geography is ignored. Parties are neither born nor do they disappear due to a lack of support.

You have just completed an important part of box 1 -- the identification of variables affecting the situation. The length of your list, nearly twenty items, illustrates the necessity of starting models simply. It would be confusing and fruitless trying to combine all of these variables on a first attempt. Incremental improvements and analyses are more revealing and practical.

If the simplifications of a model seem extreme or the results seem implausible, we try to reduce our simplifying assumptions so as to improve the model's usefulness, consistency, and believability. For the next iteration of the Representation arrow in the modeling cycle, we pick a variable that we feel should be considered prior to relying on the model's conclusions. We then extend our model by incorporating a mathematical version of our real-world variable. For example, the last two suggestions above offer refinements to the model that address the stated concern. That was a great thing to do. This process of complicating a simple model until it better reflects reality is at the heart of modeling. Let's carry out one of these modified simulations.

On an overhead, the class and I use a penny and a nickel and award tie votes (equidistant voters) to the nickel. The effect is comparable to the no overlap rule. *In pairs, please pick one or more ideas from the list and describe how the model would have to change to address the critique. State any rule changes clearly and try running your new simulation.* Student variations include:

Get data on the distribution of voters and weight each circle according to the popularity of that position.

Punish waffling – parties can only move a certain number of times.

Multiple issues could be addressed through a network with links between 1 and 4, 2 and 8 showing similarities and joint interests across the spectrum. (How then would we compute distance?)

Could also have an n-dimensional grid. Each axis could represent an issue or set of issues (abortion, domestic policy, etc.)

Several of these ideas are tried out during the next class using *Design Your Own Democracy*, a program (available for both PCs and Macintoshes from Meaningful Mathematics at http://www.meaningfulmath.org/modeling – see the program guide, **P1.4**, for directions on using the program) that simulates this model and allows the students to change various rules of the simulation and observe the resultant behavior of the parties. In a few class periods, students are able to work through several cycles of critiquing and refining a model as they generate more realistic results. All students are able to create representations appropriate to their own mathematical sophistication. It is difficult to provide an experience with so many iterations of the modeling cycle with other models that have complicated symbolic representations. An exciting aspect of this unit is the students' surprise that mathematics can be used both to describe and to offer new understandings about political and philosophical issues such as voter behavior and the design of democracies.

Creating New Representations

Representing missing variables effectively in the model requires students to practice looking for structures that both the nonmathematical setting and the mathematical objects have in common. Consider the suggestion to use a bigger marker for the incumbent (the one receiving the most votes on the previous turn) and awarding it the full vote in cases of equal voter preference. This change in the model represents the advantage held by a party in power. For example, in the real world, parties win additional votes because of the benefits (greater visibility, patronage, control of redistricting) that accrue to incumbency; analogously, in the model, the larger marker would now win extra votes because of the rule suggested by the student. The result of incumbency in both systems — real and modeled — is that weaker parties could not win by taking positions identical to those of their opponents. This process of developing an *isomorphism*, a mapping between two ideas with matching structures and parallel behaviors but often separate origins, is rarely experienced by high school students.

Modeling provides a context for teaching about isomorphism, which is one of the most powerful acts of both applied and pure mathematical thinking. Students do not always attend to the most relevant features of a variable when creating representations. They need to ask themselves what characteristics are actually being captured by their proposals. For example, using a larger marker for incumbents proved helpful because the marker's size served as a reminder of its status, but the size itself did not affect the rules of the simulation. The marker could just as easily have been a different color, and the simulation would still have provided an isomorphic representation of incumbency. In contrast, a different student proposed representing an incumbent party with a dollar bill, which would cover two circles at a time, while coins would stand for the other parties. Although this proposal also contains a satisfying iconography, it is mathematically distinct from the first. The student's playful idea not only helps keep track of which party won the previous turn but it also changes the geometry and the behavior of the simulation.

Evaluating Representations

Students practice inventing mathematical representations for many of the new variables that the class identified. For each proposed change, they need to consider whether to alter the simulation's physical design or one of its rules or both. They then try out the simulation with their changes and consider the meaning of the new behaviors that they observe.

How do students create new representations? By considering whether they are altering a behavior of the parties or voters or a structural aspect of the political world. By experimenting with changes and observing the effects of their and their classmates' proposals. By struggling to produce a clear description of their change. A modification is not valid if no one can actually carry out the simulation. As the course continues, repeated examination and alteration of models in a range of settings and involving a variety of mathematical ideas enhances students' instincts about how to best capture a real-world idea abstractly.

Student attempts to incorporate multiple political issues (in response to the concern that the model represents a linear political spectrum) are particularly interesting. Some students suggest adding links between different voters, creating a network rather than a line. Others propose a grid of two or more dimensions in which each axis would represent an issue or a set of issues. Figure 2, below, shows both proposals as well as questions that the students have raised. How do students choose between these alternative representations? They do so by paying attention to the consequences and coherence of each possibility. Both of these new configurations move beyond the geometric boundaries of a linear political spectrum. However, the first tangled network does not yield a clear mapping to the real-world situation. The new connections do make it possible for parties to jump farther, but students do not come up with an interpretation of this change that shows how it represents combinations of views on multiple political issues.

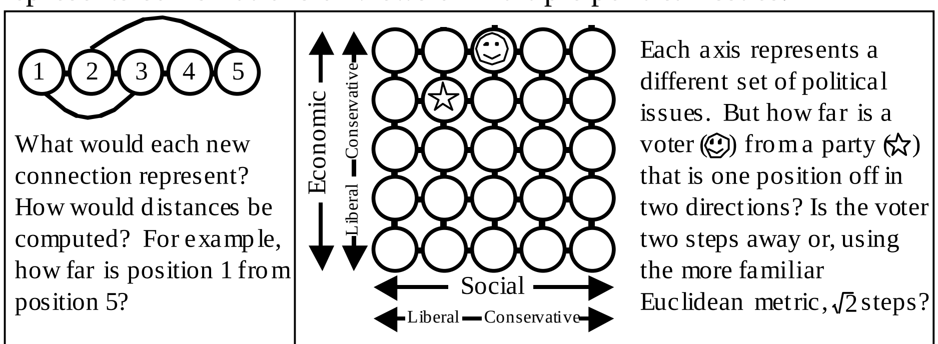

Figure 2. Student proposals for a nonlinear representation of political beliefs.

By contrast, the two-dimensional grid succeeds in representing every possible pairing of views, but students still need to make sense of how voters would make their choices (how distances between voters' and parties' positions would be measured). Should there be diagonals in the grid model? An informal discussion of metrics leads the students to compare beeline (Euclidean) and gridline (taxicab) distances.[3] They wonder whether the metric is a crucial modeling choice influencing party behavior. These considerations illustrate the rich questions about pure mathematics that can arise naturally from the complex issues and new directions suggested by modeling problems. Ultimately, students decide that the grid provides a consistent and improved representation for the model. Only through testing and by insisting that each part

[3] A *metric* is a function that determines the distance between points in a space. The familiar metric for Euclidean space is given by D = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ (which is just the Pythagorean Theorem). The taxicab metric, so-called because taxicabs in a city cannot travel diagonally through buildings, adds the horizontal and vertical changes between two points to get D = $\sqrt{|x_1 - x_2| + |y_1 - y_2|}$.

of a representation be interpreted (mapped onto a real-world characteristic of the problem) are they sure that they have made a good decision.

Students can also gauge the value of a model by looking at the predictions they can make from it. Do the mathematical results of the model translate into outcomes that are known to be true but that were not intentionally built into the representation? If so, then modelers should have an increased willingness to trust other newly discovered implications of the model that appear to have a degree of plausibility. For example, students who believe that real political parties move toward the political center in the way that the simulated parties did are more comfortable accepting other interpretations drawn from the model.

My Favorite Model Analysis

> "In and out of power, both parties tended to drift toward what their leaders described as the center, where they believed most of the voters were located. But the center was an illusion. Rather than being the midpoint of controversy, established in open debate, it represented an ideological vacuum, and the locus of inaction and indifference."
>
> - Robert Shogun on turn of the century (c. 1900) politics.[4]

Students propose a variety of strategies for modifying the political parties model to represent less-than-universal voter participation. They suggest (1) making random selections of which voters participate at each turn, (2) weighting the circles to relfect research on voter participation according to political beliefs, and (3) counting only those voters within a set distance from each party. (For example, voters at positions 10 and 11 might abstain from voting if the nearest party were at position 6.) By implementing each of these three proposals, students learn that some representations are more informative even when they may not appear to be the most realistic of the choices. The third suggestion proved to be a particularly revealing option. If voters will only support parties within three units of their own position, then the markers that represent the two parties end up at positions 4 and 8. Thus, blocs of voters who consistently withhold votes from parties that stray too far from the bloc can pull parties away from the center and assure distinct political choices. In a two-party system, voter behavior can achieve an advantage that having a greater number of parties offers in other democratic models. Thus, it is not solely how a democracy is designed, but also how people participate within the system, that determines the outcomes of the political process. Exploring this model helped the class and me focus on this different aspect of the question. The process of identifying new variables and asking, "What if the model is changed this way…" led to the insights that follow.

The voter behaviors that lead to non-converging party platforms require that people reject the lesser-of-two-evils approach to candidate choice. Voters must support only those candidacies

[4] Shogun, Robert. *None of the Above*. New York: New American Library Books, 1982.

that, to some substantial extent, reflect their outlook. This none-of-the-above approach will, in the long, run, force parties to hew to the electorate that they represent. The caveat, however, is that the party must know that the voters are actively withholding their votes and are not just apathetic citizens. The 1980s and 1990s provide an example of these behaviors at work. Organized groups of Christian conservatives made it clear that they would only support reasonably kindred candidates. A lack of enthusiasm is said to have contributed to President Bush's failed re-election in 1992. In contrast, there have been no effective groups on the left that have been willing to risk losing an election to make a point for future campaigns. Labor and environmental groups, for example, consistently support Democratic candidates who ignore labor and environmental issues just a little less than their opponents. As a result, President Clinton, who made his name leading a group trying to push the Democratic party to more centrist platforms, was able to enact a number of changes which would have been strictly Republican turf only a few years earlier (e.g., welfare reduction, increased prison penalties, civil liberties erosions, a "Don't Ask, Don't Tell" policy that led to many times more discharges from the military than took place under Presidents Reagan or Bush, a push to balance the budget instead of spend on social priorities, etc.).

Around election time, I have seen public service campaigns with the slogan "Your Vote Is Your Voice." Aside from the scary proposition that there are not other important avenues for involved citizenship, the above analysis suggests that a withheld vote complemented by an active, public declaration of dissent can be at least as effective in swaying the political process. One way to facilitate this process, and achieve the benefits of multi-party European style systems without the additional parties, would be to add a None-Of-The-Above choice to every election. If NOTA won a plurality of the vote, then new elections would be held with new candidates. I believe voter turnout would swell (thus silencing the pundits who decry an apathetic electorate) given this option.

American democracy is based on winner-take-all elections. In such a case, it is difficult for non-centrist parties to survive (in fact, they are accused of being spoilers for whatever major party is nearest their position). In other democratic systems, voter choice develops more readily. For example, in many European countries, legislative representation is assigned proportional to the vote total in an election. Thus, a Green party with 10% of the vote would be awarded 10% of the legislative seats. These parties then form majority coalitions with larger parties in order to gain control of the government. Minor parties can influence policy because their membership may be critical to a coalition's majority. Proportional systems are therefore more likely to encourage a large number of functioning parties (See Michael Lind's *Alice Doesn't Vote Here Anymore* for an excellent discussion of this issue).

<u>Teaching tips for this activity</u>

Students can get additional practice creating and testing modifications of the model as a homework assignment. Have them pick one change and write a short report which describes the real-world variable which they added to the model, the nature of the change to the model, a sample run of the simulation, the general end results of several runs of the new simulation (looking at spectrum length, party number, starting position, etc. as possible variables), and conclusions which may be drawn from their model about the political world. Their writing should be sufficiently clear that any reader could run the simulation himself or herself. Well-written and thought out reports should be shared with the entire class. Have them critique what made the report effective (if they think it was effective – I find students to be more critical than I usually am) and evaluate the reasonableness of both the representation of the new variable and the conclusions that were drawn. The student assessments can come through a class discussion or the students can read the reports and write "teacherly" feedback to the authors. Written responses are improved if the class first lists standards for constructive feedback.

Students can research the actual behaviors of political parties and their long-term evolution. How, for instance, do parties know where they are relative to the citizenry and which policies to adopt? The mathematics of the polling involved is an interesting follow-up topic. Similarly, students can test the consequences of different voting mechanisms (e.g. run-offs, preference voting, Borda counts, etc.) on voter and party behavior (see COMAP's For All Practical Purposes). Student might research different countries' political systems, their constitutions, and their laws on elections and evaluate which lead to more fair, effective, and desirable political environments.

Students working with the *Design Your Own Democracy* program can efficiently explore the questions that arise when they first change the number of parties in the simulation. They can systematically alter the length of the political spectrum and the number of parties and look for patterns in the stability or periodicity of the final party positions. They can move the parties before beginning the simulation in order to test the effect of different initial configurations on the end states. Depending on the other settings (simulation rules), start position can have an impact on either the overall behavior (where the parties end up) or individual consequences (who gets which votes). Such pure math explorations frequently arise once modeling projects begin. They help a modeler better appreciate the strengths and implications of a model. However, modelers should not necessarily assume that each result conveys a new truth about the real world. Initial position explorations can be related to sensitivity analyses, which are part of the process of assessing the dependability of a model's predictions (see Sensitivity Analysis in the Functions chapter).

Students are likely to trusting any conclusions from a model as simple as the one presented. Their instincts should be respected, but they should also be encouraged (especially after brainstorming the numerous missing variables) to note the necessity of starting any modeling project more simply than may be ultimately needed. Creating a model with dozens of variables and considerations would be too confusing. A benefit of the incremental approach is that a modeler can test each new change and discover whether the newly included relationship or variable is indeed significant in its affect on the outcomes of the model. For example, changing the voter distribution from one that is uniform to a central one does pull the parties toward the center, but no unexpected or new behaviors seem to reveal themselves. The two-dimensional model makes pleasing and intriguing patterns that have lead to fun pure math investigations, but it has not yet yielded many new real-world insights. Surprisingly often, a relatively simple model may be sufficient to inspire fresh thinking and novel, yet believable, predictions. Lastly, a simpler model is more general and may be able to answer a broader range of questions (or serve as a launching point for doing so). More complex models are quite specific to their setting and less easily applied to related problems.

Teaching the Modeling Cycle to Middle School Students

Younger students will most likely lack the political background needed for the above activity. However, the modeling cycle can be introduced with a different, yet analogous, setting (**P1.3**). The setting is now a city with a single street[5]. The entities are lunchtime customers, in the buildings on each block, and hot dog vendors, who set up their carts at the start of each day. The vendors are limited in their movements by a permit system or, perhaps, for fear of losing loyal customers. As with the political party model, the desires of the customers for easy access and of the vendors for maximum profit may be at odds. Having two vendors on the same central block makes many customers walk much farther than if the vendors were placed at the one-fourth and three-fourths marks of the street. Students should consider what, from the customers' perspectives, would be an optimal arrangement of *n* vendors[6]. This activity can demonstrate that the often-celebrated "invisible hand of the market" does not necessarily serve society's needs in an optimal fashion (if two vendors were regulated such that they had to separate their locations symmetrically, their profits would be unaffected, but the public would have greater convenience[7]).

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5 This description is intentionally extreme to encourage the critique that we really want a two-dimensional array.

6 Interestingly, certain types of businesses tend to cluster (e.g., lighting stores in lower Manhattan and car dealerships on any number of routes throughout the country) and this clustering may or may not be to everyone's mutual advantage. The development of clusters and monopolies is discussed in "Positive Feedbacks in the Economy" by Brian Arthur, Scientific American (February 1990): pp. 92-99.

7 Of course, a good walk after eating a hot dog might not be a bad idea.

Much of the political party discussion can be mapped onto this new setting. A monopolistic vendor could sit wherever they wanted. Just as parties may have varying degrees of influence or appeal, vendors might have broader offerings or better food. Vendors, like candidates, may have charisma. Where your officemate goes may influence you in the way that lawn signs influence voters.

Appendix A - Bibliography

The Modeling Cycle

Hotelling, Harold. “Stability in Competition.” The Economic Journal 39 (1929): 41-57. *This article introduces the linear city model that is an ancestor of the model of political parties competing along a linear spectrum.*

Lind, Michael. “Alice Doesn’t Vote Here Any More.” Mother Jones (March/April 1998): 53-71.

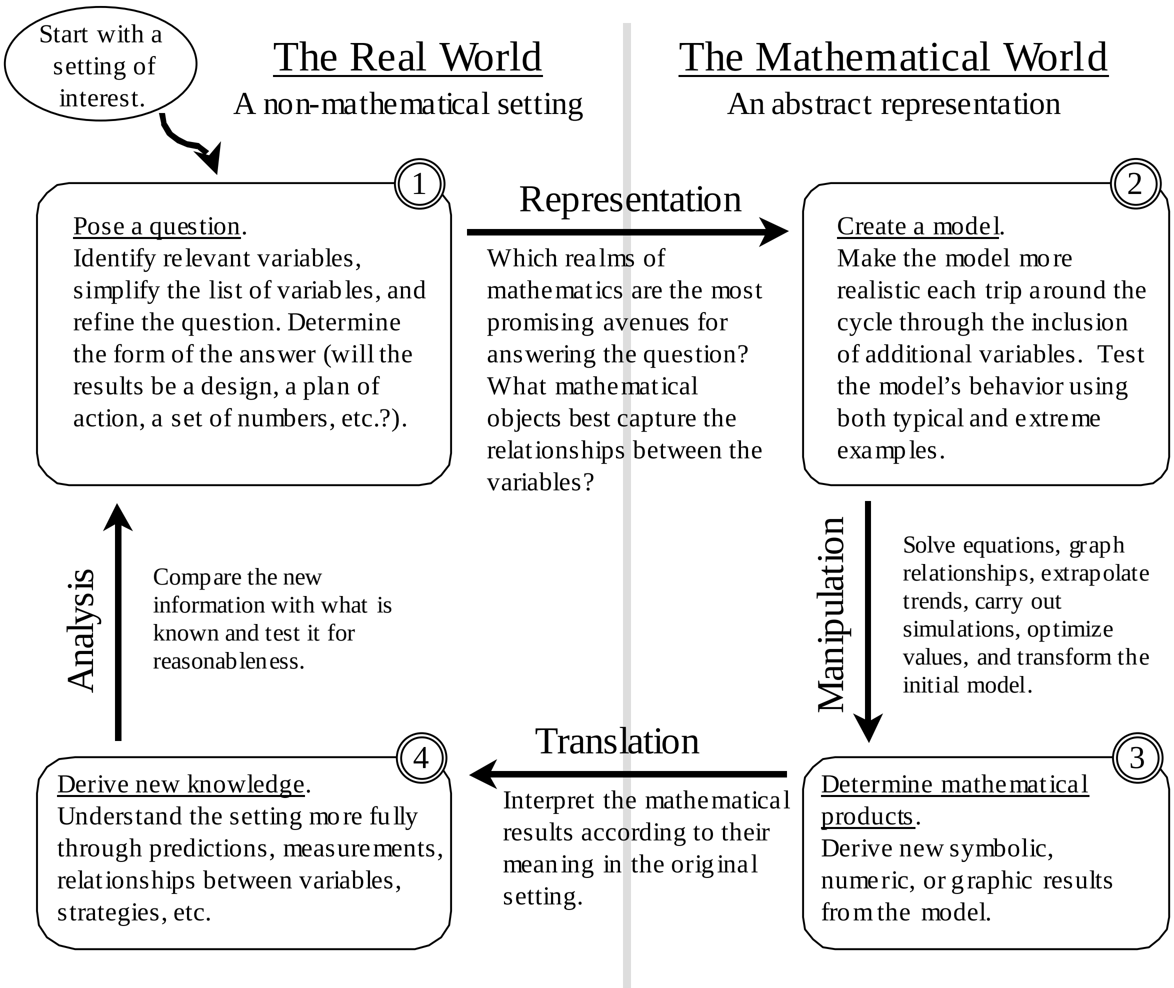
Pollak, Henry. “The Process of Applying Mathematics.” In A Sourcebook of Applications of School Mathematics, prepared by A Joint Committee of the MAA and the NCTM, pp. 1-9. Reston, Va.: National Council of Teachers Of Mathematics, 1980. *Pollak’s essay discusses the modeling cycle using extensions of Hotelling’s model. It is was the inspiration for the political parties activity.*

COMAP. For All Practical Purposes, 4th ed. W. H. Freeman, New York, 1997. This text, designed as a liberal arts introduction to mathematics for college students but which is accessible to high school students, has several fine chapters on voting methods and apportionment problems which raise additional questions about how we structure our democratic processes and institutions. COMAP has also packaged some of this material in a more abridged form for high school teaching.

<u>Appendix B - Handouts</u>*

P1.1	-	Modeling cycle diagram.
P1.2	-	Party Grid and Rules.
P1.3	-	Hot Dog Grid and Rules.
P1.4	-	*Democracy* Program Guide.

* Documents are numbered within each unit and sub-unit (N3.2 is the second document of the third sub-unit of the Numbers in Context unit).


Start with a setting of interest.
The Real World
A non-mathematical setting
The Mathematical World
An abstract representation
1
Pose a question.
Identify relevant variables, simplify the list of variables, and refine the question. Determine the form of the answer (will the results be a design, a plan of action, a set of numbers, etc.?).
Representation
Which realms of mathematics are the most promising avenues for answering the question? What mathematical objects best capture the relationships between the variables?
2
Create a model.
Make the model more realistic each trip around the cycle through the inclusion of additional variables. Test the model's behavior using both typical and extreme examples.
Manipulation
Solve equations, graph relationships, extrapolate trends, carry out simulations, optimize values, and transform the initial model.
3
Determine mathematical products.
Derive new symbolic, numeric, or graphic results from the model.
Translation
Interpret the mathematical results according to their meaning in the original setting.
4
Derive new knowledge.
Understand the setting more fully through predictions, measurements, relationships between variables, strategies, etc.
Analysis
Compare the new information with what is known and test it for reasonableness.

<table>
<tr><td colspan="2">1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10 — 11</td></tr>
<tr><td colspan="2"><u>Simulation Rules:</u></td></tr>
<tr><td><u>The Voters</u> (circles on the number line)

1. Each circle represents one voter.

2. Voters vote for the party nearest to them. If more than one party is nearest a voter, then the voter will split her vote (e.g., a three-way tie yields a third of a vote for each party).</td><td><u>The Parties</u> (coins or markers)

1. Parties take turns moving, and they can move one step to the left, one to the right, or stay still.

2. They will choose from the three options according to which option provides them with the greatest number of votes. A party will move only if its share of the vote is increased.</td></tr>
</table>

<table>
<tr><td colspan="2">1 2 3 4 5 6 7 8 9 10 11</td></tr>
<tr><td colspan="2"><u>Simulation Rules:</u></td></tr>
<tr><td><u>The Consumers</u> (circles on the number line)

1. Each circle represents one consumer.

2. Consumers buy lunch from the vendor nearest to them. If more than one vendor is nearest a consumer, then the consumer will split her purchases (e.g., a three-way tie yields a third of a purchase for each party).</td><td><u>The Vendors</u> (coins or markers)

1. Vendors take turns moving, and they can move one block to the left, one to the right, or stay still.

2. They will choose from the three options according to which option provides them with the greatest number of sales. A vendor will move only if its number of sales is increased.</td></tr>
</table>

Design Your Own…

Democracy

Thoughtful Software © 1995
Designed and Programmed by
Joshua Paul Abrams

Design Your Own Democracy accompanies the Introduction to the Modeling Cycle chapter of the book ***Mathematical Modeling: Teaching the Open-ended Application of Mathematics***. To find out more about this modeling curriculum, please visit www.meaningfulmath.org. You can contact Joshua Abrams at meaningfulmath@meaningfulmath.org

Thanks to Jeffrey Westall for his help with this documentation and for testing the software. Special thanks to Jacob Gagnon for creating the PC version of the program.

This documentation describes each of the features of the program and suggests connections between the features and the class discussion of the political parties model. If you are using Vying Vendors (intended for middle school students), the commands have new names, but the program functions identically.

The Menu Commands

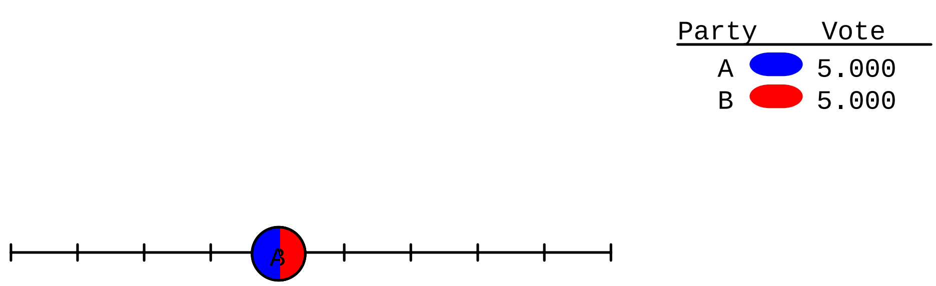
File Commands

New	This command returns you to the original one-dimensional view with the initial settings. The new document is titled “untitled”.
Open...	Opens a previously saved Democracy file.
Save	This allows you to save the current settings and the current positions of the parties.
Save As...	This command is the same as save but you are allowed to specify a new file name.
Print...	This command prints the current window view.
Quit	The program quits.

Edit Commands (Macintosh version only)

Most of these commands are disabled when the program window is active.

Copy — A picture of the current political situation is copied into the clipboard. Example:



Length Commands

4 - 14 — This list of numbers allows you to select the length of the political spectrum. This length is equivalent to the number of different positions that a party can take on a given issue. The length affects the number of voters and is equal to the number of voters in the one-dimensional, even distribution set-up. The length determines both dimensions in the two-dimensional set-up.

Parties Commands

2 - 8 — This list allows you to select the number of parties that will compete in the democracy. Students can explore how many parties are needed under different versions of the simulation before the parties spread out along the political spectrum.

Dimension Commands

1 - 2 — This menu allows you to choose a one- or two-dimensional political universe. It is recommended that the user understand the dynamics of the one-dimensional setting before moving to the two-dimensional grid. All two-dimensional grids are square. The two-dimensional grid is one possible approach to incorporating multiple political issues. Each axis can represent a different issue or set of issues. There is no natural origin or special point of intersection between the axes. The grid produces the $(\text{length})^2$ different combinations of the two positions that voters and parties might have on each issue. With the even distribution setting, there is one voter at each point of intersection of the grid. Parties in the two-dimensional model will consider their four neighboring positions (due north, south, east, and west) before moving. They do not move diagonally in a single turn.

Run Commands (In the PC version, the first four items appear as buttons on the main screen)

NOTE: At any time, the location of parties can be changed by positioning the cursor over a party, clicking, and dragging. If two parties are in the same place, the hand will grab the party whose label is later in the alphabet. The hand cursor can be used to change the initial configuration of the parties prior to running the simulation. It can be used to move a party in the middle of a simulation. It is also interesting to move a party once the parties have settled down into a stable or periodic configuration. Do slight shifts lead to an upheaval or are the parties quickly drawn back to their prior stable positions? In order to understand why a party chose a particular spot, stop the simulation and then move the party to each of its neighboring positions. If Display Election Totals is on then you can observe the effect that each move has on the party's vote total.

Go - The voters begin to vote (as fast as your processor can run the simulation).

Step - A party takes its turn and then the simulation stops. If **All Parties at Once** is checked, then one round of voting occurs and the simulation stops. Since the simulation runs rather quickly, a better understanding of the development of the parties' positions can be had by using the keyboard commands (open-apple-T for the Macintosh or "t" (once the simulation has been run once) for PCs) repeatedly to watch the parties take their turns.

Stop - If the simulation is running, this command halts the democratic process enabling you to study the current state of affairs or to adjust the conditions (menu settings can be changed and parties can be moved).

Reset - This commands returns the parties to their initial positions.

One at a Time- The parties take turns moving.

All Parties at Once - The parties plot their strategies and move simultaneously.

Vote Commands

Display Election Totals – A chart displays how much of the vote each party receives in the current arrangement.

Show How They Voted - Each intersection on the grid is colored to show which party the voter at that intersection voted for. The intersection will be colored black in case of a tie and left blank if the voter abstained from voting.

Metric - A 'metric' is a function for assigning distances to a space. It must satisfy the following properties:
$d(a, b)$ = the distance between points a and b.
$d(a, b) = d(b, a) \geq 0$.
$d(a, b) + d(b, c) \geq d(a, c)$ {the triangle inequality}
Different metrics give a space different geometric behaviors. The two metrics below are interchangeable in one dimension. However, in the two-dimensional simulation, they lead to different results. They reflect voter perception of a party's disagreement with their own views. Is a separation by one step for two separate issues better, equal to, or worse than matching on one issue and differing by two steps for the other issue?

Euclidean - Distance is measured along the diagonal. $\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}$

Taxicab - Distance is measured only along horizontal and vertical lines.

$$|x_b - x_a| + |y_b - y_a|$$

Tie Votes

Are Split - If more than one party is closest to a given voter then the voter will evenly split their vote (e.g., 1/3 of a vote each to three parties each two units away).

Go to the "Older" Party - If more than one party is closest to a given voter, the voter will cast their vote for the party whose label is earliest in the alphabet. Alphabetically earlier labels correspond to better-known, more influential parties.

Voter Distribution:

Even - Each intersection of the grid is home to one voter.

Central - More voters reside in the center of the grid than in the middle. The outermost positions are assigned one voter and then each position closer to the center receives one additional voter. For example, a nine position linear (one-dimensional) spectrum will have positions with the following numbers of voters: 1, 2, 3, 4, 5, 4, 3, 2, 1. Can students determine the pattern?

Vote If Within - Voters will only consider a party within a set number of units from them. This setting is affected by the prevailing metric. Voters with no party within this range remain neutral. This option allows voters to "punish" parties who stray too far from a given position. It simulates principled voting. It is not a good simulation of low voter turnout, which may not always be based on principles.

1 unit - 10 units - Voters will only vote for a party within this range.

Any Distance - Voters will vote for the nearest party regardless of its distance. This option is the initial setting.